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CANONICAL FORMS OF QUATERNARY ABELIAN SUBSTITUTIONS IN AN ARBITRARY GALOIS FIELD*

BY

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§1. *Introduction.*

For application to the problem of the distribution of the substitutions of a given group into complete sets of conjugates within the group, a set of canonical forms for its substitutions should have the property that two substitutions are conjugate within the group if, and only if, they are reducible to the same canonical form according to a definite scheme of reduction. In particular, if the canonical form belongs to a higher field than the initial $GF[p^n]$, the new indices introduced must be conjugate with respect to the initial field.

In the present paper is given a set of canonical forms of quaternary abelian substitutions in the $GF[p^n]$ such that the canonical forms likewise belong to the special abelian group $SA(4, p^n)$, the reduction being effected within the group. From them are derived the ultimate canonical forms, not all belonging to the given abelian group. In the former case, the canonical forms depend on the *coefficients* of the characteristic equation, in the latter case upon its *roots*.

When the given group is the *general* linear homogeneous group on m indices with coefficients in the $GF[p^n]$, a set of ultimate canonical forms is furnished by a theorem due (for the case $n = 1$) to JORDAN.† Likewise for the group of ternary linear homogeneous substitutions of determinant unity in the $GF[p^n]$, a complete set of ultimate canonical forms has been determined. ‡ The problem has also been solved for the corresponding binary group. The corresponding problem for a linear group of special character (i. e., not directly related to the general linear group) has not been previously solved so far as is known to the writer. The simplicity of the canonical substitutions for the quaternary abelian group makes comparatively easy the classification of abelian substitutions into

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† *Traité des substitutions*, pp. 114–126. A simple proof by induction of the general theorem has been given by the writer in the American Journal of Mathematics, vol. 22, p. 121, 1900.

‡ DICKSON, American Journal of Mathematics, vol. 22, p. 231, 1900.

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sets of conjugates within the abelian group (§ 23). The analogous problem is then solved (§ 25) for the simple quotient-group $A(4, p^n)$ and the results are discussed for the case $p^n = 3$, which leads to a simple group of order 25920 of frequent occurrence in geometrical problems (§§ 26–27). In addition to the checks mentioned in §§ 23, 24 upon the calculations of the paper, it may be stated that the results for the case $p^n = 3$ were previously derived by methods independent of those employed in this paper.*

Frequent use will be made of the theorem † that the equation

$$a\xi^2 + \beta\eta^2 = 1$$

has in the $GF[p^n]$ $(p > 2)$, $p^n - \nu$ solutions (ξ, η), where ν denotes $+1$ or -1 according as $-a\beta$ is a square or a not-square in the field.

§ 2. *Definition of the abelian group.*

The quaternary special abelian group $SA(4, p^n)$ is composed of the linear substitutions

$$(1) \qquad S: \quad \begin{array}{c|cccc} & \xi_1 & \eta_1 & \xi_2 & \eta_2 \\ \hline \xi_1' = & \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \eta_1' = & \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \xi_2' = & \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \eta_2' = & \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22} \end{array}$$

with coefficients in the $GF[p^n]$ which satisfy the relations ‡

$$(2) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{11} & \delta_{11} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{12} & \delta_{12} \end{vmatrix} = 1, \qquad \begin{vmatrix} \alpha_{21} & \gamma_{21} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{22} & \gamma_{22} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 1,$$

$$(3) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \alpha_{21} & \gamma_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \alpha_{22} & \gamma_{22} \end{vmatrix} = 0, \qquad \begin{vmatrix} \beta_{11} & \delta_{11} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \beta_{12} & \delta_{12} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 0,$$

$$(4) \qquad \begin{vmatrix} \alpha_{11} & \gamma_{11} \\ \beta_{21} & \delta_{21} \end{vmatrix} + \begin{vmatrix} \alpha_{12} & \gamma_{12} \\ \beta_{22} & \delta_{22} \end{vmatrix} = 0, \qquad \begin{vmatrix} \beta_{11} & \delta_{11} \\ \alpha_{21} & \gamma_{21} \end{vmatrix} + \begin{vmatrix} \beta_{12} & \delta_{12} \\ \alpha_{22} & \gamma_{22} \end{vmatrix} = 0,$$

and equivalent relations (2′), (3′), (4′), formed from the columns of (1) as the former were formed from its rows.

* An account of these elementary methods, sufficient for the case $p^n = 3$, was presented January 7, 1901, to the London Mathematical Society.

† Compare American Journal of Mathematics, vol. 21, p. 195, 1899.

‡ For $n = 1$, the abelian group was studied by JORDAN, *Traité des substitutions*, pp. 171–179; for general n, it was investigated by the writer, Quarterly Journal of Mathematics, vol. 29, pp. 169–178, 1897; vol. 31, pp. 383–4, 1899.

Of the simplest substitutions satisfying these relations, the following are frequently employed in this paper, the notations being standard: †

$$
\begin{aligned}
M_i &: \quad \xi_i' = \eta_i, \quad \eta_i' = -\xi_i;\\
L_{i\lambda} &: \quad \xi_i' = \xi_i + \lambda\eta_i;\\
L'_{i\lambda} &: \quad \eta_i' = \eta_i + \lambda\xi_i;\\
T_{i\lambda} &: \quad \xi_i' = \lambda\xi_i, \quad \eta_i' = \lambda^{-1}\eta_i;\\
P_{12} &: \quad \xi_1' = \xi_2, \quad \eta_1' = \eta_2, \quad \xi_2' = \xi_1, \quad \eta_2' = \eta_1;\\
N_{ij\lambda} &: \quad \xi_i' = \xi_i + \lambda\eta_j, \quad \xi_j' = \xi_j + \lambda\eta_i.
\end{aligned}
$$

The order N of $SA(4, p^n)$ is $p^{4n}(p^{4n}-1)(p^{2n}-1)$.

Since the general substitution (1) may be derived from the generators $L_{i\lambda}$, M_i and $N_{ij\lambda}$, its determinant is unity.

The reciprocal of S, given by (1), is

$$
(5) \qquad S^{-1}: \quad \begin{pmatrix} \delta_{11} & -\gamma_{11} & \delta_{21} & -\gamma_{21} \\ -\beta_{11} & \alpha_{11} & -\beta_{21} & \alpha_{21} \\ \delta_{12} & -\gamma_{12} & \delta_{22} & -\gamma_{22} \\ -\beta_{12} & \alpha_{12} & -\beta_{22} & \alpha_{22} \end{pmatrix}.
$$

It follows that the first minors (taken without prefixed sign) of $\alpha_{ij}, \delta_{ij}, \beta_{ij}, \gamma_{ij}$ are respectively $\delta_{ij}, \alpha_{ij}, \gamma_{ij}, \beta_{ij}$.

§ 3. *Characteristic equation of an abelian substitution.*

By definition, the characteristic determinant of S is

$$
\Delta(\kappa) \equiv \begin{vmatrix} \alpha_{11}-\kappa & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ \beta_{11} & \delta_{11}-\kappa & \beta_{12} & \delta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22}-\kappa & \gamma_{22} \\ \beta_{21} & \delta_{21} & \beta_{22} & \delta_{22}-\kappa \end{vmatrix}.
$$

The constant term $\Delta(0)$ of $\Delta(\kappa)$ expanded according to powers of κ is unity, being the determinant of the substitution. The coefficient of $-\kappa$ is

$$
\begin{vmatrix} \delta_{11} & \beta_{12} & \delta_{12} \\ \gamma_{21} & \alpha_{22} & \gamma_{22} \\ \delta_{21} & \beta_{22} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \gamma_{12} \\ \alpha_{21} & \alpha_{22} & \gamma_{22} \\ \beta_{21} & \beta_{22} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \gamma_{11} & \gamma_{12} \\ \beta_{11} & \delta_{11} & \delta_{12} \\ \beta_{21} & \delta_{21} & \delta_{22} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} \\ \beta_{11} & \delta_{11} & \beta_{12} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} \end{vmatrix}
$$

$$
= \delta_{11} + \alpha_{11} + \delta_{22} + \alpha_{22}:
$$

† If an index is not altered, it is not written in the formula.

The coefficient of κ^2 is the sum of six determinants of the second order. In particular, $\Delta(\kappa)$ has the form

$$\kappa^4 - (a_{11} + \delta_{11} + a_{22} + \delta_{22})\kappa^3 + (\ )\kappa^2 - (a_{11} + \delta_{11} + a_{22} + \delta_{22})\kappa + 1 .$$

Hence the reciprocal of any root of $\Delta(\kappa) = 0$ is itself a root. *The characteristic equation of a special abelian substitution is a reciprocal equation.**

§ 4. *Substitutions whose characteristic equations have all their roots in the $GF[p^n]$, no root being ± 1.*

Suppose first that all the roots of $\Delta(\kappa) = 0$ belong to the $GF[p^n]$. Designate them by κ, κ^{-1}, λ, λ^{-1} and consider first the case in which no root is ± 1. The root κ leads to a linear function $\omega \equiv a\xi_1 + b\eta_1 + c\xi_2 + d\eta_2$ which S multiplies by κ. But $SA(4, p^n)$ contains a substitution V which replaces ξ_1 by ω. Then $V^{-1}SV \equiv S_1$ replaces ξ_1 by $\kappa\xi_1$. Likewise the root κ^{-1}, which is also a root of the characteristic equation for S_1, leads to a linear function

$$\omega_1 \equiv a_1\xi_1 + b_1\eta_1 + c_1\xi_2 + d_1\eta_2 ,$$

which S_1 multiplies by κ^{-1}.

If $b_1 \neq 0$, the group contains the abelian substitution

$$U \equiv \begin{pmatrix} b_1^{-1} & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & d_1 \\ -b_1^{-2}d_1 & 0 & b_1^{-1} & 0 \\ c_1 & 0 & 0 & b_1 \end{pmatrix} .$$

Then $U^{-1}S_1U$, being abelian, takes the form

$$\begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \end{pmatrix} .$$

From the assumption concerning $\Delta(\kappa)$, the equation

$$\begin{vmatrix} \alpha - \kappa & \gamma \\ \beta & \delta - \kappa \end{vmatrix} = 0$$

has as its roots the distinct marks λ, λ^{-1} of the $GF[p^n]$. Hence the given substitution S is conjugate with the canonical form

* The theorem is true for any number $2m$ of indices. For proof, the direct method of the text may be employed; another proof may be based upon the canonical forms of linear substitutions in a Galois field. In a subsequent paper the writer intends to extend the present investigation to the case $2m > 4$.

$$(6) \qquad \xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = \lambda\xi_2, \quad \eta_2' = \lambda^{-1}\eta_2.$$

We take $\lambda \neq \kappa$ or κ^{-1}, the contrary cases leading to the type (7) below. Here κ has $p^n - 3$ and λ has $p^n - 5$ values if $p > 2$; while, for $p = 2$, κ has $2^n - 2$ and λ has $2^n - 4$ values. But the substitution (6) is transformed by M_1 into a similar one with κ and κ^{-1} interchanged; by M_2 into one with λ and λ^{-1} interchanged; by P_{12} into one with κ and λ, κ^{-1} and λ^{-1} interchanged. The eight resulting combinations give all the substitutions of the type (6) with the distinct roots $\kappa, \kappa^{-1}, \lambda, \lambda^{-1}$. The number of types of canonical forms is therefore

$$\tfrac{1}{8}(p^n - 3)(p^n - 5), \text{ for } p > 2; \quad \tfrac{1}{8}(2^n - 2)(2^n - 4), \text{ for } p = 2.$$

The most general substitution of $SA(4, p^n)$ commutative with a given substitution (6) has the form

$$\xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad \xi_2' = b\xi_2, \quad \eta_2' = b^{-1}\eta_2.$$

Their number being $(p^n - 1)^2$, it follows that each substitution (6) is one of $N \div (p^n - 1)^2 \equiv p^{4n}(p^{2n} + 1)(p^n + 1)^2$ conjugates within $SA(4, p^n)$.

If, however, $b_1 = 0$ in ω_1, we may suppose that $c_1 \neq 0$. For, if $c_1 = 0$, $d_1 \neq 0$, then $M_2^{-1}S_1M_2$ multiplies $a_1\xi_1 + d_1\xi_2$ by κ^{-1}. With $c_1 \neq 0$, the group contains the abelian substitution

$$V = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -a_1c_1^{-1} \\ a_1 & 0 & c_1 & d_1 \\ 0 & 0 & 0 & c_1^{-1} \end{Vmatrix}.$$

Hence $V^{-1}S_1V$ belongs to the group and has the form

$$\begin{Vmatrix} \kappa & 0 & 0 & 0 \\ \beta_{11} & \kappa^{-1} & \beta_{12} & 0 \\ 0 & 0 & \kappa^{-1} & 0 \\ \beta_{21} & 0 & \beta_{22} & \kappa \end{Vmatrix} \qquad (\beta_{21} = \kappa^2\beta_{12}).$$

Transforming by $L_{1\tau}'L_{2\sigma}'$ and taking

$$\beta_{11} + \tau(\kappa - \kappa^{-1}) = 0, \quad \beta_{22} + \sigma(\kappa^{-1} - \kappa) = 0,$$

we find a substitution of the same form, having $\beta_{11} = \beta_{22} = 0$. If $\beta_{12} = 0$, we have the canonical form

$$(7) \qquad \xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = \kappa^{-1}\xi_2, \quad \eta_2' = \kappa\eta_2.$$

If $\beta_{12} \neq 0$, we transform by $T_{2\,\beta_{12}\kappa}$ and obtain the type

$$(8) \qquad \begin{Vmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & \kappa^{-1} & 0 \\ 0 & 0 & \kappa^{-1} & 0 \\ \kappa & 0 & 0 & \kappa \end{Vmatrix}.$$

The number of non-conjugate substitutions (7) with $\kappa^2 \neq 1$ is $\frac{1}{2}(p^n - 3)$ if $p > 2$ and $\frac{1}{2}(2^n - 2)$ if $p = 2$. A substitution commutative with (7) has the form

$$\begin{Vmatrix} a & 0 & 0 & \beta \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ \gamma & 0 & 0 & \delta \end{Vmatrix}.$$

The abelian relations give

$$a\alpha - \beta b = 1, \quad ac - \beta d = 0, \quad -\gamma a + \delta b = 0, \quad -\gamma c + \delta d = 1.$$

Letting $\Delta \equiv ad - bc$, we have, as the solution of these relations,

$$\alpha = d/\Delta, \quad \beta = c/\Delta, \quad \gamma = b/\Delta, \quad \delta = a/\Delta.$$

The number of the commutative substitutions is therefore $(p^{2n} - 1)(p^{2n} - p^n)$, so that each substitution (7) is conjugate within $SA(4, p^n)$ with exactly $p^{3n}(p^{2n} + 1)(p^n + 1)$ substitutions.

The substitution (8) is transformed by P_{12} into a similar one with κ^{-1} in place of κ. Hence there are $\frac{1}{2}(p^n - 3)$ or $\frac{1}{2}(2^n - 2)$ non-conjugate substitutions (8) with $\kappa^2 \neq 1$. A linear substitution commutative with (8) has the form

$$\begin{Vmatrix} a & 0 & 0 & 0 \\ 0 & c & d & 0 \\ 0 & 0 & c & 0 \\ b & 0 & 0 & a \end{Vmatrix}.$$

The abelian relations give $ac = 1$, $-bc + ad = 0$, whence

$$c = a^{-1}, \quad d = bc^2.$$

The number of such substitutions is therefore $p^n(p^n - 1)$. Hence each substitution (8) is one of $p^{3n}(p^{4n} - 1)(p^n + 1)$ conjugates.

§ 5. *Roots in the field, two of them being* ± 1.

Suppose next that the roots are $\kappa, \kappa^{-1}, \pm 1, \pm 1$, where κ is a mark $\neq 0$ or ± 1 of the $GF[p^n]$. The canonical form is

$$(9) \qquad \begin{Vmatrix} \kappa & 0 & 0 & 0 \\ 0 & \kappa^{-1} & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \beta & \pm 1 \end{Vmatrix}.$$

For $\beta = 0$ there are $p^n - 3$ or $\frac{1}{2}(2^n - 2)$ types each commutative with

$$(10) \qquad \begin{Vmatrix} t & 0 & 0 & 0 \\ 0 & t^{-1} & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{Vmatrix} \qquad (ad - bc = 1),$$

giving $(p^n - 1)p^n(p^{2n} - 1)$ substitutions. Hence each type yields a set of $p^{3n}(p^{2n} + 1)(p^n + 1)$ conjugate substitutions of $SA(4, p^n)$.

For $\beta \neq 0$, the substitutions with β a square are transformed into each other by abelian substitutions $T_{2\sigma}$ and likewise those with β a not-square. The two sets are seen to be not conjugate within $SA(4, p^n)$. The number of types is therefore $2(p^n - 3)$ or $\frac{1}{2}(2^n - 2)$ according as $p > 2$ or $p = 2$. A substitution commutative with (9) for $\beta \neq 0$, $\kappa^2 \neq 1$ has the form (10) with $b = 0$ and $a = d$. Each type is therefore commutative with $2p^n(p^n - 1)$ or $2^n(2^n - 1)$ substitutions and thus conjugate with exactly $\frac{1}{2}p^{3n}(p^{4n} - 1)(p^n + 1)$ or $2^{3n}(2^{4n} - 1)(2^n + 1)$ substitutions within $SA(4, p^n)$.

§ 6. *Two roots each* ± 1 *and two roots each* ∓ 1.

If the roots are $\pm 1, \pm 1, \mp 1, \mp 1$, the canonical form is

$$(11) \qquad \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ a & \pm 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ 0 & 0 & \beta & \mp 1 \end{Vmatrix}.$$

One may chose the lower sign, transforming if necessary by P_{12}.

For $a = \beta = 0$, the substitution becomes T_{1-1} and, for $p > 2$, is commutative only with the $[p^n(p^{2n} - 1)]^2$ substitutions

$$(12) \qquad \begin{Vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \end{Vmatrix} \qquad (ad - bc = 1,\ a_1 d_1 - b_1 c_1 = 1).$$

For $a \neq 0$, we may suppose that $a = 1$ or ν, where ν is a particular not-square in the $GF[p^n]$; indeed, a is replaced by $a\sigma^{-2}$ upon transforming (11) by $T_{1\sigma}$. Similarly, we may assume that $\beta = 0$, 1 or ν. The resulting eight types are:

$$L_{1\mu}T_{1-1}, \quad L_{2\mu}T_{1-1}, \quad L_{11}T_{1-1}, \quad L_{2\mu}, \quad L_{1\nu}T_{1-1}L_{2\mu}$$

where $\mu = 1$ or ν. The number of conjugates to each may be determined directly or more simply by the method of §§ 9–10.

§ 7. *Four equal roots each* ± 1.

If the roots be $\pm 1, \pm 1, \pm 1, \pm 1$, the canonical form is either

$$L \equiv \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ a & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & \beta & \pm 1 \end{bmatrix} \quad \text{or} \quad R \equiv \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & \beta_{12} & \delta_{12} \\ a_{21} & 0 & \pm 1 & 0 \\ \beta_{21} & 0 & a_{22} & \pm 1 \end{bmatrix}$$

according as the linear function ω_1 determined by the second root ± 1 contains η_1 or does not. If $\beta_{12} = \delta_{12} = 0$, R is of the form L. If $\delta_{12} = 0$, $\beta_{12} \neq 0$, the transform of R by $T_{2\beta_{12}}$ is of the form R with $\beta_{12} = 1$, $\delta_{12} = 0$. The abelian relations give $a_{21} = 0$, $\beta_{21} = 1$. Then, for $\beta_{11} \neq 0$, we transform by the abelian substitution

$$\xi_1' = \xi_1 + \beta_{11}^{-1}\xi_2, \quad \eta_2' = \eta_2 - \beta_{11}^{-1}\eta_1,$$

and reach a substitution of the form L. A similar result follows if $\beta_{11} = 0$, $a_{22} \neq 0$, since the transform of R by P_{12} then has $\beta_{11} \neq 0$. Finally, if $\beta_{11} = a_{22} = 0$, we have the substitution

$$\begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 1 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 1 & 0 & 0 & \pm 1 \end{bmatrix}. \tag{13}$$

For $p > 2$, this is transformed into L (with $a = 1$, $\beta = -1$) by

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 1 \\ -1 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

For $p = 2$, substitution (13) will be seen to furnish a new type. A substitution (1) commutative with (13) must have the form

$$\begin{Vmatrix} \alpha_{11} & 0 & \alpha_{12} & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ \delta_{12} & 0 & \delta_{11} & 0 \\ \beta_{21} & \alpha_{12} & \beta_{22} & \alpha_{11} \end{Vmatrix},$$

subject to the abelian conditions

$$\alpha_{11}\delta_{11} + \alpha_{12}\delta_{12} = 1, \quad \beta_{11}\alpha_{12} + \delta_{11}\beta_{21} + \alpha_{11}\beta_{12} + \delta_{12}\beta_{22} = 0.$$

By the former, α_{11} and α_{12} are not both zero, so that the latter determines one of the β_{ij} in terms of the other three, which may be chosen arbitrarily in the $GF[2^n]$. Hence there are $2^n(2^{2n} - 1)2^{3n}$ substitutions of $SA(4, 2^n)$ commutative with (13). A substitution L is conjugate with the identity or L_{11} or $L_{11}L_{21}$. By § 10, $L_{11}L_{21}$ is commutative with exactly 2^{4n} substitutions of $SA(4, 2^n)$, so that $L_{11}L_{21}$ and (13) are not conjugate within the group. A different argument is necessary for the case of L_{11} and (13); but the latter are readily shown to be not conjugate under abelian transformation.

It remains to consider R when $\delta_{12} \neq 0$. The transformed of R by $T_{2\,\delta_{12}^{-1}}$ is of the form R with $\delta_{12} = 1$. The latter is transformed by $L'_{2\,\beta_{12}}$ into the substitution

$$R_1 = \begin{Vmatrix} \pm 1 & 0 & 0 & 0 \\ \beta_{11} & \pm 1 & 0 & 1 \\ -1 & 0 & \pm 1 & 0 \\ \mp a & 0 & a & \pm 1 \end{Vmatrix}.$$

Suppose first that $p = 2$. If $a = 0$, the transform of R_1 by M_2 is of the form R with $\delta_{12} = 0$, a case previously considered. If $a \neq 0$, the transform of R_1 by $T_{1\kappa}T_{2\kappa}$ is of the form R_1 with $\kappa^{-2}a$ in place of a. Choosing $\kappa = a^{\frac{1}{3}}$, we obtain a substitution

$$R_\beta = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ \beta & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{Vmatrix}$$

of period 4. Transforming R_β by the abelian substitution

$$\xi'_1 = \xi_1, \quad \eta'_1 = \eta_1 + \tau\eta_2 + \tau^2\xi_2, \quad \xi'_2 = \xi_2 - \tau\xi_1, \quad \eta'_2 = \eta_2 + \tau\xi_2,$$

we obtain, for $p = 2$, the substitution R_b, where $b \equiv \beta + \tau + \tau^2$. In order

that the $GF[2^n]$ shall contain a mark τ for which $b = 0$, it is necessary and sufficient that

$$B \equiv \beta + \beta^2 + \beta^4 + \cdots + \beta^{2^{n-1}} = 0 .$$

Since β belongs to the field, $B^2 = B$. Inversely, there are 2^{n-1} marks β making $B = 0$ and as many making $B = 1$. The 2^{n-1} substitutions R_β for which $B = 0$ are therefore conjugate within $SA(4, 2^n)$. Likewise the 2^{n-1} substitutions R_β for which $B = 1$ are all conjugate; indeed, R_β is conjugate with R_b and $b \equiv \beta + \tau + \tau^2$ takes 2^{n-1} distinct values when τ runs through the series of 2^n marks, while

$$b + b^2 + b^4 + \cdots + b^{2^{n-1}} \equiv \beta + \beta^2 + \beta^4 + \cdots + \beta^{2^{n-1}} + \tau + \tau^{2^n} \equiv B \pmod 2.$$

That the substitutions R_β for which $B = 1$ are not conjugate with R_0 may be shown by considering the condition $R_\beta S = S R_0$, S being of the general form (1). We find that S must have the form

$$(14) \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \beta_{11} & 1 & \beta_{12} & \beta_{22} \\ \beta_{22} & 0 & 1 & 0 \\ \beta_{21} & 0 & \beta_{22} & 1 \end{bmatrix} \qquad \begin{pmatrix} \beta_{21} = \beta_{12} + \beta_{22} + \beta \\ \beta_{21} = \beta_{12} + \beta_{22}^2 \end{pmatrix}.$$

The latter conditions require that $\beta + \beta_{22} + \beta_{22}^2 \equiv 0 \pmod 2$.

For $p = 2$, the only substitutions of $SA(4, 2^n)$ commutative with R_β are of the form (14) subject, however, to the conditions

$$\beta_{21} = \beta_{12} + \beta_{22}^2, \quad \beta_{21} = \beta_{12} + \beta_{22} .$$

Hence $\beta_{22} = 0$ or 1, while β_{11} and β_{12} are arbitrary; thus there are $2 \cdot 2^{2n}$ substitutions.

For $p > 2$, R_1 is transformed into a similar substitution R' having $\beta_{11} = 0$ by the following abelian substitution

$$\eta_1' = \eta_1 + \tfrac{1}{2}\beta_{11}\xi_2, \quad \eta_2' = \eta_2 + \tfrac{1}{2}\beta_{11}\xi_1 .$$

For $a = 0$, the transform of R' by M_2 is of the form (13).

For $a \neq 0$, the transform of R' by $T_{1,\pm 1}$ gives the substitutions

$$A_a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ -a & 0 & a & 1 \end{bmatrix}, \qquad A_{-a} T_{1,-1} T_{2,-1} .$$

Transforming A_a by $T_{1,\kappa} T_{2,\kappa}$ we obtain $A_{a\kappa^{-2}}$. Hence, if $p > 2$, there are only four canonical types, ν being a particular not-square:

$$(15) \qquad A_1,\ A_\nu,\ A_1T_{1,-1}T_{2,-1},\ A_\nu T_{1,-1}T_{2,-1}.$$

The substitutions A_μ and $A_\mu T_{1,-1}T_{2,-1}$ are not conjugate, if $p>2$, their characteristic equations having different roots. But every substitution commutative with one is commutative with the other, $T_{1,-1}T_{2,-1}$ being commutative with every quaternary linear substitution. The period of A_μ is readily seen to be p if $p>3$, 9 if $p=3$, or 4 if $p=2$.* Then $A_\mu T_{1,-1}T_{2,-1}$ is of period $2p$ if $p>3$, 18 if $p=3$, or 4 if $p=2$.

If S be the general substitution (1), the identity $A_\alpha S = SA_{\alpha'}$ requires that

$$\gamma_{11}=\gamma_{12}=\gamma_{21}=\gamma_{22}=a_{12}=\delta_{21}=0,\quad a_{22}=a_{11},\quad \delta_{22}=\delta_{11},\quad \beta_{22}=a\delta_{12},$$

$$\beta_{21}=-\beta_{12}-a\delta_{12},\quad a\delta_{22}=a'a_{22},\quad -a'a_{11}+a'a_{21}=-\beta_{22}-a\delta_{22}.$$

The second abelian relation (2) then gives $a_{22}\delta_{22}=1$, whence

$$a\delta_{22}^2=a'.$$

Hence A_α and $A_{\alpha'}$ are conjugate within $SA(4, p^n)$ only when a and a' are both squares or both not-squares in the $GF[p^n]$. To determine the substitutions S commutative with A_α, set $a'=a$. Then

$$S=\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ \beta_{11} & a_{11} & \beta_{12} & \delta_{12} \\ -\delta_{12} & 0 & a_{11} & 0 \\ -\beta_{12}-a\delta_{12} & 0 & a\delta_{12} & a_{11} \end{bmatrix},$$

subject to the abelian relations

$$a_{11}^2=1,\quad 2a_{11}\beta_{12}+aa_{11}\delta_{12}-a\delta_{12}^2=0.$$

For each of the two values of a_{11}, the second relation determines δ_{12}. Hence there are $2p^{2n}$ substitutions commutative with A_α. Hence each of the substitutions (15) is conjugate with exactly $\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$ substitutions within $SA(4, p^n)$, while no two of the four canonical types are conjugate.

§ 8. *Study of the abelian substitutions of type L.*

Upon transforming L by $T_{1\sigma}T_{2\kappa}$, we obtain a substitution of the form L with a, β replaced by $a\sigma^{-2}$, $\beta\kappa^{-2}$. Hence the substitutions L are conjugate with one of the following:

* For example, by introducing the new indices

$$X=-\xi_1,\ Y=\xi_2,\ Z=\xi_2-a^{-1}\eta_2,\ W=-a^{-1}\eta_1-\xi_2+a^{-1}\eta_2,$$

A_α takes the standard canonical form (not abelian):

$$X'=X,\ Y'=Y+X,\ Z'=Z+Y,\ W'=W+Z.$$

$$L_{1\mu}L_{2\tau}, \quad L_{1\mu}L_{2\tau}T_{1-1}T_{2-1} \qquad (\mu, \tau = 0, 1, \nu).$$

Of these, $L_{11}L_{2\nu}$ is transformed into $L_{1\nu}L_{21}$ by P_{12}, and $L_{11}L_{21}$ is transformed into $L_{1\nu}L_{2\nu}$ by the abelian substitution

$$\begin{bmatrix} \nu\delta & 0 & \nu\sigma & 0 \\ 0 & \delta & 0 & \sigma \\ -\nu\sigma & 0 & \nu\delta & 0 \\ 0 & -\sigma & 0 & \delta \end{bmatrix},$$

subject only to the condition $\nu(\delta^2 + \sigma^2) = 1$, which has solutions in every $GF[p^n]$. The identity and $T_{1-1}T_{2-1}$ are conjugate only with themselves. Hence the types L remaining for consideration are:

$$L_{1\mu}, \quad L_{1\mu}T_{1-1}T_{2-1}, \quad L_{1\mu}L_{21}, \quad L_{1\mu}L_{21}T_{1-1}T_{2-1} \qquad (\mu = 1, \nu).$$

The characteristic equations for the second and fourth substitutions have the roots -1 and hence are not conjugate with the first or third.

The substitutions L_{11} and $L_{1\nu}$, ν being a not-square in the $GF[p^n]$, $p > 2$, are not conjugate within the $SA(4, p^n)$. In fact, $SL_{11} = L_{1\nu}S$ gives the conditions:

$$\beta_{11} = 0, \quad \delta_{11} = \nu\alpha_{11}, \quad \beta_{12} = \beta_{21} = \alpha_{21} = \delta_{12} = 0.$$

Thus S does not satisfy the abelian relation (2):

$$1 = \alpha_{11}\delta_{11} - \beta_{11}\gamma_{11} + \alpha_{12}\delta_{12} - \beta_{12}\gamma_{12} \equiv \nu\alpha_{11}^2.$$

The fact that $L_{11}L_{21}$ and $L_{1\nu}L_{21}$ are not conjugate with each other and that neither is conjugate with either L_{11} or $L_{1\nu}$ within $SA(4, p^n)$ follows incidentally from the following determination of the number of abelian substitutions conjugate with each of the four. To determine the number of substitutions of $SA(4, p^n)$ conjugate with $L_{1\mu}L_{2\tau}$, let S denote the general substitution (1) commutative with it. The conditions for the identity

$$SL_{1\mu}L_{2\tau} = L_{1\mu}L_{2\tau}S \qquad (\mu \neq 0)$$

are found at once to be the following:

$$\beta_{11} = 0, \quad \beta_{12} = 0, \quad \beta_{21} = 0, \quad \alpha_{11} = \delta_{11}, \quad \tau\beta_{22} = 0,$$
$$\mu\alpha_{21} = \tau\delta_{21}, \quad \mu\delta_{12} = \tau\alpha_{12}, \quad \tau\alpha_{22} = \tau\delta_{22}.$$

For $\tau = 0$, S has the form

$$\begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & \alpha_{11} & 0 & 0 \\ 0 & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & \delta_{21} & \beta_{22} & \delta_{22} \end{bmatrix}. \tag{16}$$

In particular, an abelian relation gives $a_{11}^2 = 1$. Hence $S = Y$ or $T_{1,-1}Y$, where Y is the most general substitution of $SA(4, p^n)$ which leaves η_1 fixed. The number of substitutions Y is $p^{3n}p^n(p^{2n}-1)$, being equal to the number of substitutions of $SA(4, p^n)$ which leave fixed the index ξ_1.* According as $p > 2$ or $p = 2$, $L_{1\mu}$ is one of $\frac{1}{2}(p^{4n}-1)$ or $2^{4n}-1$ conjugate substitutions within $SA(4, p^n)$.

For $\tau = 1$, a substitution S commutative with $L_{1\mu}L_{21}$ has the form

$$(17) \qquad \begin{bmatrix} a_{11} & \gamma_{11} & \mu\delta_{12} & \gamma_{12} \\ 0 & a_{11} & 0 & \delta_{12} \\ a_{21} & \gamma_{21} & a_{22} & \gamma_{22} \\ 0 & \mu a_{21} & 0 & a_{22} \end{bmatrix},$$

subject to the abelian relations

$$a_{11}^2 + \mu\delta_{12}^2 = 1, \quad a_{22}^2 + \mu a_{21}^2 = 1, \quad a_{11}^2 + \mu a_{21}^2 = 1,$$

$$a_{11}a_{21} + \delta_{12}a_{22} = 0, \quad a_{11}\gamma_{21} - \gamma_{11}a_{21} + \mu\delta_{12}\gamma_{22} - \gamma_{12}a_{22} = 0.$$

Hence

$$\delta_{12}^2 = a_{21}^2, \quad a_{22} = \pm a_{11}, \quad a_{11}(a_{21} \pm \delta_{12}) = 0.$$

Suppose first that μ is a not-square ν in the $GF[p^n]$, $p > 2$. Then $a_{11} \neq 0$ and $a_{21} = \mp\delta_{12}$, $a_{22} = \pm a_{11}$. For any one of the $p^n + \epsilon$ sets of solutions of $a_{11}^2 + \nu\delta_{12}^2 = 1$, ϵ being ± 1 according as $p^n = 4l \pm 1$, a_{21} and a_{22} are determined except in sign; while γ_{21} is determined in terms of $\gamma_{11}, \gamma_{12}, \gamma_{22}$. Hence there are $2p^{3n}(p^n + \epsilon)$ substitutions of $SA(4, p^n)$ commutative with $L_{1\nu}L_{21}$.

Suppose next that $\mu = 1$. Whether a_{11} be zero or not, we may set $a_{22} = \pm a_{11}$, $a_{21} = \mp\delta_{12}$. For any one of the $p^n - \epsilon$ sets of solutions of $a_{11}^2 + \delta_{12}^2 = 1$ in the $GF[p^n]$, $p > 2$, a_{22} and a_{21} are determined except in sign, and one of the γ_{ij} is determined in terms of the remaining three. Hence there are $2p^{3n}(p^n - \epsilon)$ substitutions of $SA(4, p^n)$, $p > 2$, commutative with $L_{11}L_{21}$. For $p = 2$, we get $a_{22} = a_{11}$, $\delta_{12} = a_{21}$, $a_{11} + \delta_{12} = 1$, so that $SA(4, 2^n)$ contains exactly 2^{4n} substitutions commutative with $L_{11}L_{21}$.

§ 9. *Study of the substitutions* $L_{1\mu}T_{1,-1}$ *and* $L_{1\mu}T_{2,-1}$ ($\mu = 1$ *or* ν).

If ν be a not-square in the $GF[p^n]$, $p > 2$, no two of the substitutions $L_{11}T_{1,-1}$, $L_{1\nu}T_{1,-1}$, $L_{11}T_{2,-1}$, $L_{1\nu}T_{2,-1}$ are conjugate within $SA(4, p^n)$. The first two are not conjugate and the last two are not conjugate since their $(p+1)$-th powers are L_{11} and $L_{1\nu}$ and are not conjugate by § 8. Finally, a relation $SL_{1\mu}T_{1,-1} = L_{1\tau}T_{2,-1}S$ is proved impossible by forming the respective products.

*Quarterly Journal of Mathematics, vol. 29, pp. 171–173, 1897.

Within $SA(4, p^n)$, $p > 2$, each of the four substitutions is one of a complete set of $\frac{1}{2}p^{2n}(p^{4n}-1)$ conjugate substitutions. In proof, let S be an abelian substitution commutative with $L_{1\mu}T_{1-1}$. Then S is commutative with the p-th and $(p+1)$-th powers of the latter, which are T_{1-1} and $L_{1\mu}$ respectively. Hence S is at the same time of the forms (12) and (16). Hence

$$S = \begin{bmatrix} a_{11} & \gamma_{11} & 0 & 0 \\ 0 & a_{11} & 0 & 0 \\ 0 & 0 & a_{22} & \gamma_{22} \\ 0 & 0 & \beta_{22} & \delta_{22} \end{bmatrix},$$

subject to the abelian relations

$$a_{11}^2 = 1, \quad a_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1.$$

The number of substitutions S is therefore $2p^n p^n(p^{2n}-1)$.

A substitution S' commutative with $L_{1\mu}T_{2-1}$ will be commutative with $L_{1\mu}T_{2-1}\cdot T_{1-1}T_{2-1} \equiv L_{1\mu}T_{1-1}$ and vice versa. Hence every S' is an S and vice versa. Hence the final theorem:

Within $SA(4, p^n)$, $p > 2$, the substitutions $L_{11}T_{1-1}$, $L_{11}T_{2-1}$, $L_{1\nu}T_{1-1}$ and $L_{1\nu}T_{2-1}$, where ν is a not-square in the field, are not conjugate and each gives rise to a complete set of $\frac{1}{2}p^{2n}(p^{4n}-1)$ conjugate substitutions.

§ 10. *Study of the substitutions $L_{1\mu}T_{1-1}L_{2\tau}$ ($\mu, \tau = 1$ or ν).*

The substitutions $L_{11}T_{1-1}L_{21}$ and $L_{1\nu}T_{1-1}L_{21}$ are not conjugate within $SA(4, p^n)$, $p > 2$. Indeed, their $p+1$-th powers, $L_{11}L_{21}$ and $L_{1\nu}L_{21}$ are not conjugate by § 8. Likewise $L_{11}T_{1-1}L_{2\nu}$ and $L_{1\nu}T_{1-1}L_{2\nu}$ are not conjugate within $SA(4, p^n)$. Finally, $L_{1\mu}T_{1-1}L_{21}$ is not conjugate with $L_{1\kappa}T_{1-1}L_{2\nu}$; for, if S transform the former into the latter, it is seen that S must replace ξ_2 by $\nu\delta_{22}\xi_2 + \gamma_{22}\eta_2$ and η_2 by $\delta_{22}\xi_2$, where $\nu\delta_{22}^2 = 1$. Combining the four non-conjugate substitutions into the single type $E \equiv L_{1\mu}T_{1-1}L_{2\tau}$, where $\mu, \tau = 1, \nu$, it will be shown that each E is commutative with exactly $4p^{2n}$ substitutions of $SA(4, p^n)$, $p > 2$, and hence is one of a set of $\frac{1}{4}(p^{4n}-1)(p^{2n}-1)p^{2n}$ conjugate substitutions. In proof, let S be commutative with E. Then S must be commutative with $E^p = T_{1-1}$, so that S must be of the form (12). Also S must be commutative with $E^{p+1} = L_{1\mu}L_{2\tau}$ and hence have $\beta_{11} = 0$, $\beta_{22} = 0$, $a_{11} = \delta_{11}$, $a_{22} = \delta_{22}$ (§ 8). Hence S has the form

$$\begin{bmatrix} a & \gamma & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a_{22} & \gamma_{22} \\ 0 & 0 & 0 & a_{22} \end{bmatrix} \qquad (a^2 = a_{22}^2 = 1).$$

Inversely, each of these substitutions is evidently commutative with E.

In view of the number of the substitutions of $SA(4, p^n)$ commutative with the substitutions E and the number commutative with the four of § 9, none of the latter are conjugate with a substitution E.

§ 11. *Canonical form of a binary substitution of irreducible characteristic determinant.*

Theorem. *A binary linear substitution in the* $GF[p^n]$

$$\Sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \qquad (\alpha\delta - \beta\gamma = 1)$$

whose characteristic determinant $D(\kappa) \equiv \kappa^2 - \kappa(\alpha + \delta) + 1$ *is irreducible in the field may be transformed into the canonical form*

$$\Sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & \alpha + \delta \end{pmatrix}$$

by a linear substitution of determinant unity and belonging to the field.

If $\beta\gamma = 0$, then $D(\kappa) = (\kappa - \alpha)(\kappa - \delta)$, contrary to hypothesis. Transforming Σ by $\begin{pmatrix} 1 & 0 \\ \alpha/\gamma & 1 \end{pmatrix}$, we obtain $S \equiv \begin{pmatrix} 0 & \gamma \\ -\gamma^{-1} & \alpha + \delta \end{pmatrix}$. If γ be a square in the field, the transform of S by $T_{1\gamma^{\frac{1}{2}}}$ gives Σ_1. If γ be a not-square, so that $p > 2$, the transform of S by $\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$ is a substitution of the form Σ with $c \equiv \gamma + \tau(\alpha + \delta) + \tau^2\gamma^{-1}$ in place of the coefficient γ. Since

$$c\gamma^{-1} = 1 + \tau\gamma^{-1}(\alpha + \delta) + (\tau\gamma^{-1})^2$$

is irreducible in the field, c cannot vanish. Moreover, at most two values of τ give the same value to the expression c. Hence c has at least $\frac{1}{2}(p^n + 1)$ values $\neq 0$, at least one of which is therefore a square in the field. By the earlier case, the substitution is conjugate with Σ_1.

By an analogous proof, Σ may be transformed into

$$\Sigma_1^{-1} \equiv \begin{pmatrix} \alpha + \delta & -1 \\ 1 & 0 \end{pmatrix}.$$

§ 12. *Substitutions whose characteristic determinant is the product of two linear factors and an irreducible quadratic factor.*

Let the characteristic determinant $\Delta(\kappa)$ of the abelian substitution S be the product of two linear factors and an irreducible quadratic factor each belonging to the $GF[p^n]$. Denote the roots of the former by a, b and those of the latter by λ, λ^{p^n}. Then (§ 3), $b = a^{-1}$, $\lambda^{p^n} = \lambda^{-1}$; hence $\lambda^{p^n+1} = 1$, $\lambda^{p^n-1} \neq 1$, the latter excluding only the two roots ± 1 of the former, so that λ may have $p^n - 1$ or 2^n distinct values, according as $p > 2$ or $p = 2$.

If $a^{-1} \neq a$, S may be transformed by a substitution of $SA(4, p^n)$ into a substitution S' which replaces ξ_1 by $a\xi_1$ and η_1 by $a^{-1}\eta_1$. On account of the abelian conditions, S' is seen to affect ξ_2, η_2 according to a substitution of the form

$$\Sigma \equiv \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \qquad (\alpha\delta - \beta\gamma = 1).$$

From the invariance of $\Delta(\kappa)$, it follows that λ, λ^{-1} are the roots of

$$\kappa^2 - \kappa(a + \delta) + 1 = 0.$$

In particular, the latter is irreducible in the $GF[p^n]$. By §11, there exists a substitution on ξ_2, η_2 with coefficients in the field and having determinant unity which transforms Σ into

$$\Sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & a + \delta \end{pmatrix}.$$

A substitution S of the group $SA[4, p^n]$ whose characteristic determinant is the product of two distinct linear factors $\kappa - a$, $\kappa - a^{-1}$ and an irreducible quadratic factor $\kappa^2 - A\kappa + 1$ is conjugate within the group with the canonical substitution

$$(18') \qquad \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A \end{bmatrix}.$$

If $a^{-1} = a$, S may be transformed by a substitution of the group into a substitution S_1 which replaces ξ_1 by $a\xi_1$ and η_1 by $a\eta_1 + b\xi_1$ and therefore ξ_2, η_2 by linear functions of ξ_2, η_2 only. If $b \neq 0$, we transform S_1 by T_{1c} and obtain a substitution of the form S_1 with $c^{-2}b$ in place of b, so that b may be restricted to unity and (for $p > 2$) a particular not-square ν of the field. *The canonical forms within $SA(4, p^n)$ of substitutions whose characteristic determinant is the product of two equal linear factors $\kappa \mp 1$ and an irreducible quadratic factor $\kappa^2 - A\kappa + 1$ are*

$$(19') \qquad \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ b & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A \end{bmatrix} \qquad (b = 0, 1 \text{ or } \nu).$$

To obtain ultimate canonical forms, we introduce the new indices, conjugate with respect to the $GF[p^n]$,

$$X \equiv -\xi_2 + \lambda\eta_2, \quad Y = -\xi_2 + \lambda^{-1}\eta_2.$$

Then (18′) and (19′) each replace X by

$$-\lambda\xi_2 + (\lambda A - 1)\eta_2 = -\lambda\xi_2 + \lambda^2\eta_2 = \lambda(-\xi_2 + \lambda\eta_2).$$

Hence (18′) and (19′) are reducible to the canonical forms

$$(18) \qquad \xi_1' = a\xi_1, \quad \eta_1' = a^{-1}\eta_1, \quad X' = \lambda X, \quad Y' = \lambda^{-1}Y \qquad (a \neq \pm 1),$$

$$(19) \qquad \xi_1' = \pm\xi_1, \quad \eta_1' = \pm\eta_1 + b\xi_1, \quad X' = \lambda X, \quad Y' = \lambda^{-1}Y.$$

Two substitutions of $SA(4, p^n)$ reducible in this manner to the same canonical form (18) or (19) are conjugate within the group and inversely.

There are $\frac{1}{4}(p^n - 1)(p^n - 3)$ or $\frac{1}{4}2^n(2^n - 2)$ non-conjugate canonical types (18). Indeed, M_1 transforms (18) into a like substitution with a and a^{-1} interchanged; M_2 transforms (18) into a like substitution with λ and λ^{-1} interchanged. A substitution of $SA(4, p^n)$ commutative with one having the canonical form (18) has simultaneously the canonical form

$$\xi_1' = c\xi_1, \quad \eta_1' = c^{-1}\eta_1, \quad X' = \tau X, \quad Y' = \tau^{-1}Y,$$

where c is any mark $\neq 0$ of the $GF[p^n]$ and τ any root of $\tau^{p^n-1} = 1$, giving $(p^n - 1)(p^n + 1)$ substitutions. Hence each type (18) represents a complete set of $p^{4n}(p^{4n} - 1)$ conjugate substitutions of $SA(4, p^n)$.

For $b = 0$, there are $p^n - 1$ or 2^{n-1} non-conjugate types (19). A substitution of $SA(4, p^n)$ commutative with one having the canonical form (19) with $b = 0$ has simultaneously the canonical form

$$(20) \qquad \xi_1' = r\xi_1 + s\eta_1, \quad \eta_1' = t\xi_1 + u\eta_1, \quad X' = \tau X, \quad Y' = \tau^{-1}Y,$$

where $ru - st = 1$, giving $p^n(p^{2n} - 1)(p^n + 1)$ substitutions. Each type thus represents $p^{3n}(p^{2n} + 1)(p^n - 1)$ conjugate substitutions of $SA(4, p^n)$.

For $b = 1$ or a not-square ν, there are altogether $2(p^n - 1)$ or 2^{n-1} non-conjugate types (19). The commutative substitutions have the canonical form (20) with $s = 0$, $r = u = \pm 1$, giving $2p^n(p^n + 1)$ or $2^n(2^n + 1)$ substitutions. Hence each type represents a set of $\frac{1}{2}p^{3n}(p^{4n} - 1)(p^n - 1)$ or $2^{3n}(2^{4n} - 1)(2^n - 1)$ conjugate substitutions.

§ 13. *Substitutions whose characteristic equations have no root in the $GF[p^n]$.*

THEOREM. *Within $SA(4, p^n)$ any substitution S, whose characteristic equation $\Delta(\kappa) = 0$ has no root in the $GF[p^n]$, is conjugate with a substitution replacing ξ_1 by $\gamma\eta_1$.*

Let S replace ξ_1 by $\omega \equiv \alpha_{11}\xi_1 + \gamma_{11}\eta_1 + \alpha_{12}\xi_2 + \gamma_{12}\eta_2$. Suppose first that $\gamma_{11} \neq 0$. There exists in $SA(4, p^n)$ a substitution T which replaces ξ_1 by $\gamma_{11}^{-1}\xi_1$ and η_1 by ω [compare § 4]. Then $T^{-1}ST$ replaces ξ_1 by $\gamma_{11}^{-1}\eta_1$. The same result will follow if S be conjugate within $SA(4, p^n)$ with a substitution having $\gamma_{11} \neq 0$. In the contrary case $\gamma_{11} = \beta_{11} = \gamma_{22} = \beta_{22} = 0$ in S and all its conjugates; indeed, by transforming S by M_1, P_{12} or $P_{12}M_1$, we may bring the

coefficient β_{11}, γ_{22} or β_{22} in the place of the former γ_{11}. By an earlier transformation, we can make $\beta_{12} = \gamma_{12} = 0$.* The resulting substitution is

$$S' \equiv \begin{Vmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & \delta_{11} & 0 & \delta_{12} \\ a_{21} & \gamma_{21} & a_{22} & 0 \\ \beta_{21} & \delta_{21} & 0 & \delta_{22} \end{Vmatrix}.$$

The transform of S' by $\eta_1' = \eta_1 + \eta_2$, $\xi_2' = \xi_2 - \xi_1$ has β_{21} as the coefficient of ξ_1 in η_1'. Hence $\beta_{21} = 0$. Similarly $\gamma_{21} = 0$. Next L_{11} transforms S' into a substitution with $\delta_{11} - a_{11}$ as the coefficient of η_1 in ξ_1'. Hence $\delta_{11} = a_{11}$ and similarly $\delta_{22} = a_{22}$. The transform of S' by $N_{12}: \xi_1' = \xi_1 + \eta_2$, $\xi_2' = \xi_2 + \eta_1$, has $\delta_{21} - a_{12}$ as the coefficient of η_1 in ξ_1'. Hence $a_{12} = \delta_{21}$ and similarly $a_{21} = \delta_{12}$. Now a_{12} or δ_{12} is not zero, since otherwise the characteristic equation has a root a_{11} or δ_{11} in the field. The transformed of S' by

$$\xi_1' = \xi_1 + \tau\xi_2, \quad \eta_2' = \eta_2 - \tau\eta_1 \qquad (a_{11} + \tau\delta_{12} = 0),$$

has the form

$$\begin{Vmatrix} 0 & 0 & a_{12}' & 0 \\ 0 & 0 & 0 & \delta_{12} \\ \delta_{12} & 0 & 0 & 0 \\ 0 & a_{12}' & 0 & 0 \end{Vmatrix}.$$

It is transformed by $T_{2a_{12}'}$ into P_{12} whose characteristic determinant is $(1 - \kappa^2)^2$, contrary to the hypothesis concerning the roots of $\Delta(\kappa) = 0$.

§ 14.

The further discussion of the resulting substitution

$$(21) \qquad \begin{Vmatrix} 0 & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & \beta_{12} & \delta_{12} \\ 0 & \gamma_{21} & a_{22} & \gamma_{22} \\ 0 & \beta_{21} & \beta_{22} & \delta_{22} \end{Vmatrix}$$

is separated into the cases $\beta_{12} = \delta_{12} = 0$, when the substitution has the form (12), and β_{12}, δ_{12} not both zero. In the latter case we may take $\beta_{12} \neq 0$, first transforming by M_2 if $\delta_{12} \neq 0$. Transforming by $T_{2\beta_{12}}$, we have a similar substitution with $\beta_{12} = 1$. Then the transform by $L_{2\delta_{12}}$ becomes

* Quarterly Journal of Mathematics, vol. 32, pp. 42-63, § 5, 1900.

$$S_1 \equiv \begin{Vmatrix} 0 & \gamma_{11} & 0 & 0 \\ \beta_{11} & \delta_{11} & 1 & 0 \\ 0 & \gamma_{11}\gamma_{22} & a_{22} & \gamma_{22} \\ 0 & \gamma_{11}\delta_{22} & \beta_{22} & \delta_{22} \end{Vmatrix} \quad (\beta_{11}\gamma_{11} = -1, \quad a_{22}\delta_{22} - \beta_{22}\gamma_{22} = 1).$$

It has the characteristic determinant

$$(22) \quad \kappa^4 - \kappa^3(\delta_{11} + a_{22} + \delta_{22}) + \kappa^2(2 + \delta_{11}a_{22} + \delta_{11}\delta_{22} - \gamma_{11}\gamma_{22}) - \kappa(\delta_{11} + a_{22} + \delta_{22}) + 1.$$

It is the product of two factors $\kappa^2 - \sigma\kappa + 1$ and $\kappa^2 - \lambda\kappa + 1$ if, and only if,

$$\sigma + \lambda = a_{22} + \delta_{22} + \delta_{11}, \quad \sigma\lambda = (a_{22} + \delta_{22})\delta_{11} - \gamma_{11}\gamma_{22}$$

may be satisfied by marks σ, λ of the $GF[p^n]$. For $p > 2$, the necessary and sufficient condition is that $(a_{22} + \delta_{22} - \delta_{11})^2 + 4\gamma_{11}\gamma_{22}$ be zero or a square in the field, viz., $(\sigma - \lambda)^2$. In particular, if $\gamma_{22} = 0$, we have

$$\Delta(\kappa) \equiv (\kappa^2 - \delta_{11}\kappa + 1)[\kappa^2 - \kappa(a_{22} + \delta_{22}) + 1].$$

§ 15.

We seek the conditions under which new indices

$$X \equiv a\xi_1 + b\eta_1 + c\xi_2 = d\eta_2, \quad Y \equiv a_1\xi_1 + b_1\eta_1 + c_1\xi_2 + d_1\eta_2$$

may be introduced so that S_1 will replace X by Y and Y by $-X + AY$, where A, a, b, c, d, a_1, $\cdots$, are marks of the $GF[p^n]$ satisfying the equation

$$(23) \quad \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} + \begin{vmatrix} c & d \\ c_1 & d_1 \end{vmatrix} = 1.$$

In order that S_1 shall replace X by Y and Y by $-X + AY$, we must have

$$\begin{cases} a_1 = b\beta_{11}, & b_1 = a\gamma_{11} + b\delta_{11} + c\gamma_{11}\gamma_{22} + d\gamma_{11}\delta_{22}, \\ c_1 = b + ca_{22} + d\beta_{22}, & d_1 = c\gamma_{22} + d\delta_{22}, \end{cases}$$

$$\begin{cases} -a = -a_1A + b_1\beta_{11}, & -b = a_1\gamma_{11} + b_1(\delta_{11} - A) + c_1\gamma_{11}\gamma_{22} + d_1\gamma_{11}\delta_{22}, \\ -c = b_1 + c_1(a_{22} - A) + d_1\beta_{22}, & -d = c_1\gamma_{22} + d_1(\delta_{22} - A). \end{cases}$$

On substitution of the values of a_1, b_1, c_1, d_1 into the second set of equations, we have the following equations:

$$(24) \quad b\beta_{11}(\delta_{11} - A) - c\gamma_{22} - d\delta_{22} = 0,$$

$$(25) \quad a(\delta_{11} - A)\gamma_{11} + b(\gamma_{11}\gamma_{22} + \delta_{11}^2 - \delta_{11}A) + c\gamma_{11}\gamma_{22}(a_{22} + \delta_{22} + \delta_{11} - A) + d\gamma_{11}(\delta_{22}^2 + \beta_{22}\gamma_{22} + \delta_{11}\delta_{22} - A\delta_{22}) = 0,$$

$$(26) \quad a\gamma_{11} + b(\delta_{11} + a_{22} - A) + c\{\gamma_{11}\gamma_{22} + a_{22}(a_{22} + \delta_{22} - A)\} + d\{\gamma_{11}\delta_{22} + \beta_{22}(a_{22} + \delta_{22} - A)\} = 0,$$

$$(27) \quad b\gamma_{22} + c\gamma_{22}(a_{22} + \delta_{22} - A) + d\delta_{22}(a_{22} + \delta_{22} - A) = 0.$$

Multiplying (24) by $a_{22}+\delta_{22}-A$ and adding to (27), we find

$$\gamma_{22}+\beta_{11}(\delta_{11}-A)(a_{22}+\delta_{22}-A)=0. \tag{28}$$

Indeed, if $b=0$, then $a_1=0$, $a=0$; then (24) and (26) require

$$a_{22}+\delta_{22}-A=0,$$

whence (24) and (25) require $d=0$, $\gamma_{22}c=0$. Then $d_1=0$, so that (23) will not hold. If we calculate the determinant of the coefficients of (24), (25), (26), (27), we find γ_{11}^2 times the square of the left member of (28). Hence the latter is the condition that these equations have solutions other than $a=b=c=d=0$.

In order that A determined by (28), viz.,

$$A^2-A(\delta_{11}+a_{22}+\delta_{22})+\delta_{11}(a_{22}+\delta_{12})-\gamma_{11}\gamma_{22}=0,$$

shall belong to the $GF[p^n]$ it is necessary and sufficient that $\Delta(\kappa)$ decomposes into quadratic factors $\kappa^2-A\kappa+1$, $\kappa^2-A'\kappa+1$ [see § 14].

For the case of $\gamma_{22}=0$, the decomposition is evident:

$$A=\delta_{11},\quad A'=a_{22}+\delta_{22}.$$

If these roots be equal, we have

$$a_{22}(a_{22}+\delta_{22}-\delta_{11})=a_{22}^2-\delta_{11}a_{22}+1=0,$$

since $a_{22}\delta_{22}=1$, so that $\kappa^2-\delta_{11}\kappa+1$ would be reducible in the field. Hence if $\Delta(\kappa)$ has equal irreducible quadratic factors, then $\gamma_{22}\neq 0$.

In case (28) is satisfied by a mark A, equation (27) may be dropped from consideration. Multiplying (24) by γ_{11} and adding to (26), we find

$$a\gamma_{11}+ba_{22}+ca_{22}(a_{22}+\delta_{22}-A)+d\beta_{22}(a_{22}+\delta_{22}-A)=0. \tag{26'}$$

Multiplying (24) by $\gamma_{11}\delta_{11}$ and (27) by $-\gamma_{11}$, and adding the resulting equations to (25), we find

$$d=a(\delta_{11}-A). \tag{25'}$$

Using the relation $1+\beta_{22}\gamma_{22}=a_{22}\delta_{22}$ and (28), we have from (26') and (24)

$$a_{22}\{a\gamma_{11}\delta_{22}+b+c(a_{22}+\delta_{22}-A)\}=0,$$

$$(\delta_{11}-A)\{a\gamma_{11}\delta_{22}+b+c(a_{22}+\delta_{22}-A)\}=0.$$

If $\delta_{11}-A=0$, then $\gamma_{22}=0$ and $a_{22}\neq 0$. Hence, in every case

$$b=-a\gamma_{11}\delta_{22}-c(a_{22}+\delta_{22}-A). \tag{29}$$

Hence, if (28) be satisfied, equations (24), (25), (26), (27) are equivalent to the two (25') and (29).

It remains to prove that the condition (23) may be satisfied. For $\gamma_{22} = 0$, we may take $A = \delta_{11}$ so that $d = 0$, $d_1 = 0$. Then

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \equiv a^2\gamma_{11} + ab\delta_{11} - b^2\beta_{11} = 1.$$

For $p = 2$, we may take $b = 0$, $a = \gamma_{11}^{\frac{1}{2}}$, every mark of the $GF[2^n]$ being a square. For $p > 2$, the condition may be written

$$(a - \tfrac{1}{2} b\beta_{11}\delta_{11})^2 + b^2\beta_{11}^2(1 - \tfrac{1}{4}\delta_{11}^2) = -\beta_{11}.$$

It has solutions a, b in the $GF[p^n]$, unless $\delta_{11} = \pm 2$. In the latter case $\kappa^2 - \delta_{11}\kappa + 1 = (\kappa \mp 1)^2$, contrary to hypothesis. Then c is determined by (29) since $a_{22} + \delta_{22} - \delta \neq 0$.

For $\gamma_{22} \neq 0$, we may suppose $\delta_{22} = 0$. Indeed the transform of S_1 by $L'_{2\tau}$, $\delta_{22} + \tau\gamma_{22} = 0$, is of the form S_1 with $\delta_{22} = 0$, $\gamma_{22} \neq 0$. Then

$$d = a(\delta_{11} - A), \quad b = -c(a_{22} - A), \quad a_1 = -c\beta_{11}(a_{22} - A),$$

$$b_1 = a\gamma_{11} - cA(a_{22} - A), \quad c_1 = a\beta_{22}(\delta_{11} - A) + cA, \quad d_1 = c\gamma_{22}.$$

Hence the condition (23) becomes

$$a^2\{\gamma_{11} - \beta_{22}(\delta_{11} - A)^2\} - acA(a_{22} + \delta_{11} - 2A) + c^2\{\gamma_{22} - \beta_{11}(a_{22} - A)^2\} = 1.$$

Since $\beta_{22}\gamma_{22} = -1$, $\gamma_{11}\gamma_{22} = (\delta_{11} - A)(a_{22} - A)$, the condition may be written

$$(a_{22} + \delta_{11} - 2A)\{a^2\gamma_{11}^2 - ac\gamma_{11}A(a_{22} - A) + c^2(a_{22} - A)^2\} = \gamma_{11}(a_{22} - A).$$

Here $a_{22} - A \neq 0$. Hence the equation is impossible if $2A = a_{22} + \delta_{11}$, whence $4\gamma_{11}\gamma_{22} + (a_{22} - \delta_{11})^2 = 0$. In this case $\Delta(\kappa)$ is the square of a quadratic factor (see § 14).

This case being excluded here the above condition may be satisfied, when $p > 2$, unless $1 - \frac{1}{4}A^2 = 0$ [cf. § 1]. Then $\gamma_{11}\gamma_{22} = (\delta_{11} \mp 2)(a_{22} \mp 2)$, so that

$$4\gamma_{11}\gamma_{22} + (a_{22} - \delta_{11})^2 \equiv (a_{22} + \delta_{11} \mp 4)^2,$$

$$\Delta(\kappa) \equiv [\kappa^2 - \kappa(a_{22} + \delta_{11} \mp 2) + 1][\kappa^2 \mp 2\kappa + 1],$$

contrary to the hypothesis concerning the roots of $\Delta(\kappa) = 0$. For $p = 2$, the condition is satisfied by the values

$$a = \gamma_{11}^{-\frac{1}{2}}A(a_{22} - A)^{\frac{1}{2}}(a_{22} + \delta_{11})^{-\frac{1}{2}}, \quad c = \gamma_{11}^{\frac{1}{2}}(a_{22} - A)^{-\frac{1}{2}}(a_{22} + \delta_{11})^{-\frac{1}{2}}.$$

As a first conclusion of the preceding investigation it follows that if the characteristic determinant $\Delta(\kappa)$ of a substitution S of $SA(4, p^n)$ decomposes in the $GF[p^n]$ into two distinct irreducible quadratic factors $\kappa^2 - A\kappa + 1$ and $\kappa^2 - A'\kappa + 1$, then S is conjugate within $SA(4, p^n)$ with a substitution of the form (12) and therefore conjugate with

$$(30) \qquad \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & A & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & A' \end{bmatrix}.$$

If $\Delta(\kappa)$ be the square of an irreducible quadratic factor, then S is either conjugate with (21) for $\beta_{12} = \delta_{12} = 0$ and hence conjugate with (30), or else is conjugate with S_1 with $\gamma_{22} \neq 0$. In the latter case we transform by $L'_{2\tau}$ and make $\delta_{22} = 0$, obtaining the substitution

$$S_2 \equiv \begin{bmatrix} 0 & \gamma_1 & 0 & 0 \\ -\gamma_1^{-1} & \delta & 1 & 0 \\ 0 & \gamma_1\gamma & a & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{bmatrix}.$$

The characteristic determinant of S_2 is

$$\Delta(\kappa) \equiv \kappa^4 - \kappa^3(\delta + a) + \kappa^2(2 + \delta a - \gamma\gamma_1) - \kappa(\delta + a) + 1.$$

In particular, $\Delta(\kappa)$ is a perfect square * if, and only if,

$$(31) \qquad -4\gamma\gamma_1 = (a - \delta)^2.$$

A second result of the investigation is that S is conjugate with a substitution of the form S_2 within $SA(4, p^n)$ when $\Delta(\kappa)$ is irreducible in the $GF[p^n]$ or is the product of two factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, belonging to and irreducible in the $GF[p^n]$. The latter cases are treated in §17; the case in which (31) holds is considered in the next section.

§16. *Characteristic determinant the square of an irreducible quadratic.*

Consider the following substitution of the form S_2:

$$(32) \qquad \Sigma_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & c & a & c \\ 0 & 0 & -c^{-1} & 0 \end{bmatrix} \qquad (-4c = a^2).$$

It has the characteristictic determinant (whether $p = 2$ or $p > 2$)

$$\kappa^4 - \kappa^3 a + \kappa^2(2 - c) - \kappa a + 1 \equiv [\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1]^2.$$

* If $p > 2$ and (31) is satisfied, $\Delta(\kappa) = \{\kappa^2 - \frac{1}{2}(a + \delta)\kappa + 1\}^2$.

A substitution S_2 satisfying (31) will have the same characteristic determinant as Σ_2 if, and only if, $a = a + \delta$, and for $p = 2$, $c = a^2 - \gamma\gamma_1$. The latter are consequently necessary conditions for the conjugacy of S_2 and Σ_2 under linear transformation. We proceed to prove that, if S_2 satisfy the condition (31) and if $\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1$ be irreducible in the $GF[p^n]$, then $a = a + \delta$, together with $c = a^2 - \gamma\gamma_1$ if $p = 2$, are sufficient conditions for the conjugacy of S_2 and Σ_2 within $SA(4, p^n)$. Assuming these conditions satisfied, we may determine a substitution S of $SA(4, p^n)$ such that $SS_2 = \Sigma_2 S$. We take for S the general substitution (1), the latter relation imposes a set of conditions which reduce to the following upon applying $a = a + \delta$, $-4c = a^2$ and (31):

$$\begin{aligned}
&\gamma_{11} = -\gamma_1\beta_{11}, \quad ca_{12} = \gamma_1\delta_{12}, \quad \gamma_{22} = -\gamma c\beta_{22}, \quad a_{21} = \gamma\delta_{21},\\
&ca_{12} = -a_{11} + \gamma_1\delta_{11}, \qquad c\gamma_1\beta_{12} = -aa_{11} + a\gamma_1\delta_{11} + c\gamma_{11} - \gamma_{12},\\
&a_{21} = \gamma_1^{-1}a_{11} - \delta_{11} - \delta\beta_{11}, \qquad \gamma_1\gamma_{21} = a\gamma_1\delta_{11} + c\gamma_{11} - aa_{11} - \gamma_{12},\\
&\gamma_1\gamma\beta_{21} = \delta a_{11} + \gamma_{12}, \qquad \gamma\gamma_1 c\beta_{22} = aa_{11} - c\gamma_{11} - a\gamma_1\delta_{11},\\
&a_{22} = \delta_{11} + a\beta_{12}, \qquad \gamma\gamma_1\delta_{22} = ca_{11} - a\gamma_{12}.
\end{aligned}$$

It suffices to take $a_{12} = 0$. Setting $\rho \equiv \gamma_1^{-1}\gamma_{12}$, we have S in the form

$$\begin{bmatrix}
\gamma_1\delta_{11} & -\gamma_1\beta_{11} & 0 & \gamma_1\rho\\
\beta_{11} & \delta_{11} & -\beta_{11} - c^{-1}\rho & 0\\
-\delta\beta_{11} & -\delta\delta_{11} - c\beta_{11} - \rho & \delta_{11} - a\beta_{11} - ac^{-1}\rho & -c\beta_{11}\\
\gamma^{-1}\delta\delta_{11} + \gamma^{-1}\rho & -\gamma^{-1}\delta\beta_{11} & \gamma^{-1}\beta_{11} & c\gamma^{-1}\delta_{11} - a\gamma^{-1}\rho
\end{bmatrix}.$$

The abelian relations (2), (3), (4) here reduce to the two:

$$\begin{aligned}
&\delta_{11}^2 + \beta_{11}^2 + \beta_{11}\rho + c^{-1}\rho^2 = \gamma_1^{-1},\\
&-\delta\delta_{11}^2 - \delta\beta_{11}^2 - c\delta_{11}\beta_{11} - 2\delta_{11}\rho + a\beta_{11}\rho + ac^{-1}\rho^2 = 0.
\end{aligned} \tag{33}$$

Multiplying the first by a and the second by -1 and adding, we have

$$(a + \delta)(\delta_{11}^2 + \beta_{11}^2) + c\delta_{11}\beta_{11} + 2\delta_{11}\rho = a\gamma_1^{-1}. \tag{34}$$

If $p > 2$, we may eliminate ρ between (33) and (34) and find

$$\delta_{11}^2\{-\beta_{11}^2(c + 4) + 8a^{-1}a\gamma_1^{-1} - 4\gamma_1^{-1}\} = \{2\beta_{11}^2 - (-c)^{-\frac{1}{2}}a\gamma_1^{-1}\}^2. \tag{35}$$

We prove that there exists a mark β_{11}^2 making the coefficient of δ_{11}^2 a square and the right member not zero. Let

$$\tau \equiv 8a^{-1}a\gamma_1^{-1} - 4\gamma_1^{-1}.$$

For $p > 2$, $\tau \neq 0$; since $2a^{-1}a = 1$ requires $a = \delta$ and therefore $\gamma\gamma_1 = 0$. If τ be square, we may take $\beta_{11} = 0$, when (35) determines $\delta_{11} \neq 0$ and (34) determines ρ. Suppose finally that τ is a not-square. Since $\kappa^2 - (-c)^{\frac{1}{2}}\kappa + 1$

is irreducible, $-c-4$ is a not-square. After we divide (35) by τ, the question is the possibility of finding a mark μ^2 such that μ^2+1 is a not-square. There are $\frac{1}{4}(p^n \mp 1)$ such squares $\mu^2 \neq 0$, according as $p^n = 4l \pm 1$.* Hence for $p^n > 5$, there are at least two marks β_{11}^2 making the coefficient of δ_{11}^2 in (35) a square, and hence at least one mark β_{11}^2 making also the right member $\neq 0$. Then δ_{11} is determined different from zero, so that (34) gives ρ.

For $p^n = 5$, the right member of (35) vanishes only if

$$\beta_{11}^2 = \pm 1, \quad (-c)^{-\frac{1}{2}} a\gamma_1^{-1} = \pm 2.$$

Then τ is a not-square only when $\gamma_1^{-1} = +1$ or -1. Since γ_1 is a square, it may be taken to be $+1$ by an earlier transformation of S_2 by $T_{1\omega}T_{2\omega^{-1}}$, ω being suitably chosen. With $\gamma_1 = 1$, then $a^{-1}a = \pm 1$. But $a^{-1}a = 1$ makes τ a square. Hence $a^{-1}a = -1$, so that $\delta = -2a$, $a = -2\delta$. Since $-c-4$ is a not-square, and $-c = \delta^2$, it follows that $\delta^2 = 1$, $a^2 = -1$. Since β_{11}^2 was chosen to make the coefficient of δ_{11}^2 a square, and since τ is now 3, it follows that $\beta_{11}^2 = -1$, and $\delta_{11} = 0$. Then (34) is an identity and (33) becomes $(2\rho - \beta_{11})^2 = 1$ and may be satisfied.

For $p^n = 3$, $c \equiv 0 \pmod 3$, since $-c-4$ is to be a not-square, necessarily -1. Hence the substitution Σ_2 cannot be employed. Since $a + \delta = 0$, (31) gives $\gamma\gamma_1 \equiv -a^2 \equiv -1 \pmod 3$. Hence S_2 takes the form

$$[\alpha, \gamma] \equiv \begin{Vmatrix} 0 & -\gamma^{-1} & 0 & 0 \\ \gamma & -a & 1 & 0 \\ 0 & -1 & a & \gamma \\ 0 & 0 & -\gamma^{-1} & 0 \end{Vmatrix}.$$

But $P_{12}M_1M_2$ transforms $[\alpha, \gamma]$ into $[-\alpha, -\gamma]$. Hence $[\alpha, \gamma]$ is conjugate either with $[1, -1]$ or with $[-1, -1]$. The latter is transformed into the former by the special abelian substitution

$$\begin{Vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{Vmatrix}.$$

For $p = 2$, we have $a = 0$, $a + \delta = 0$, $c = a^2 - \gamma\gamma_1$, so that (34) becomes $c\delta_{11}\beta_{11} = a\gamma_1^{-1}$. The latter together with (33) are to be satisfied by marks β_{11}, δ_{11}, ρ of the $GF[2^n]$. Letting $\rho = c\beta_{11}r^2$ in (33), dividing by $c\beta_{11}^2$ and extracting the square root, we find

$$r^2 + r \equiv c^{-\frac{1}{2}} + \frac{\gamma_1^{-\frac{1}{2}} + \delta_{11}}{c^{\frac{1}{2}}\beta_{11}} \pmod 2$$

$$= c^{-\frac{1}{2}} + g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})$$

*American Journal of Mathematics, vol. 21, p. 196.

upon eliminating β_{11} and denoting the constant $c^{\frac{1}{2}}a^{-1}\gamma_1$ by g. It remains to prove that a mark δ_{11} may be founded for which this equation has a root r belonging to the field. But, by § 7, $r^2 + r = \tau$ is irreducible if, and only if,

$$\tau + \tau^2 + \tau^4 + \cdots + \tau^{2^{n-1}} = 1 .$$

By hypothesis $\kappa^2 + c^{\frac{1}{2}}\kappa + 1 = 0$ and therefore $\mu^2 + \mu + c^{-1} = 0$ is irreducible in the field, so that

$$c^{-1} + (c^{-1})^2 + (c^{-1})^4 + \cdots + (c^{-1})^{2^{n-1}} = 1 .$$

If, for every mark δ_{11}, the equation in r be irreducible, then

$$g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11}) + \{g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})\}^2 + \cdots + \{g\delta_{11}(\gamma_1^{-\frac{1}{2}} + \delta_{11})\}^{2^{n-1}} \equiv 0$$

must be an identity modulo 2 in the variable δ_{11}. This will be the case if, and only if, $g = g^2\gamma_1^{-1}$, whence $g = \gamma_1$. The latter requires $c = a^2$. But $c = a^2 - \gamma\gamma_1$. Hence $\gamma = 0$, which is impossible.

In addition to the determination of the canonical form (32) and for $p^n = 3$, $[1, -1]$, we have derived the theorem:

If two substitutions S_2 have as (common) characteristic determinant the square of an irreducible quadratic, they are conjugate within the group $SA(4, p^n)$.

§ 17.

It remains to consider the cases in which $\Delta(\kappa)$ is irreducible in the $GF[p^n]$ or is the product of two irreducible quadratic factors of the form $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$. In either case the roots of $\Delta(\kappa) = 0$ are σ, σ^{-1}, σ^{p^n}, σ^{-p^n}; in the former case $\sigma^{p^{2n}+1} = 1$, in the latter case $\sigma^{p^{2n}-1} = 1$. We may write $\Delta(\kappa) = 0$ in the form

$$\sigma^2\gamma\gamma_1 = (\sigma^2 - \sigma a + 1)(\sigma^2 - \sigma\delta + 1) . \tag{36}$$

The substitution S_2 given at the end of § 15, multiplies by σ the function

$$X_1 \equiv -\gamma\xi_1 + \sigma\gamma\gamma_1\eta_1 + (\sigma^2 - \sigma\delta + 1)\xi_2 + \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1)\eta_2 .$$

Denote by Y_1, X_2, Y_2 the linear functions derived from X_1 upon replacing σ by σ^{-1}, σ^{p^n}, σ^{-p^n} respectively. If $\Delta(\kappa)$ be irreducible, so that σ belongs to the $GF[p^{4n}]$, the functions X_1, Y_1, X_2, Y_2 are conjugate with respect to the $GF[p^n]$. In the second case, σ belongs to the $GF[p^{2n}]$, so that X_1 and X_2, Y_1 and Y_2 are conjugate with respect to the $GF[p^n]$. Hence the four functions satisfy the requirements as to the conjugacy. In terms of these functions taken as new indices,* the substitution S_2 takes the canonical form

$$X_1' = \sigma X_1, \quad Y_1' = \sigma^{-1}Y_1, \quad X_2' = \sigma^{p^n}X_2, \quad Y_2' = \sigma^{-p^n}Y_2 . \tag{37}$$

* It may be verified by direct calculation that the determinant of the transformation of indices does not vanish; but the result follows from the abelian character of the transformation.

The transformation of indices satisfies those abelian conditions specified by formulæ (3) and (4). In fact,

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{p^n}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2-\sigma\delta+1 & \sigma^{-1}\gamma(\sigma^2-\sigma\delta+1) \\ \sigma^{2p^n}-\sigma^{p^n}\delta+1 & \sigma^{-p^n}\gamma(\sigma^{2p^n}-\sigma^{p^n}\delta+1) \end{vmatrix}$$

$$= \gamma(\sigma-\sigma^{p^n})\{\gamma\gamma_1+(\sigma^2-\sigma\delta+1)(\sigma^{2p^n}-\sigma^{p^n}\delta+1)\sigma^{-p^n-1}\}.$$

On elimination of $\gamma\gamma_1$ by (36), the quantity in brackets vanishes if

$$\sigma^{p^n-1}(\sigma^2-\sigma a+1)+\sigma^{2p^n}-\sigma^{p^n}\delta+1=0.$$

The latter is derived by multiplying by σ^{p^n} the identity

$$\sigma+\sigma^{-1}+\sigma^{p^n}+\sigma^{-p^n}=a+\delta,$$

which follows from the form of $\Delta(\kappa)=0$, with the roots $\sigma, \sigma^{-1}, \cdots$. In a similar manner, we find that

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{-p^n}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2-\sigma\delta+1 & \sigma^{-1}\gamma(\sigma^2-\sigma\delta+1) \\ \sigma^{-2p^n}-\sigma^{-p^n}\delta+1 & \sigma^{p^n}\gamma(\sigma^{-2p^n}-\sigma^{-p^n}\delta+1) \end{vmatrix} = 0.$$

Replacing σ by σ^{-1} in the two identities just established, we obtain two new identities. The four embrace the relations (3) and (4). Consider next the left member of the first abelian relation (2):

$$\begin{vmatrix} -\gamma & \sigma\gamma\gamma_1 \\ -\gamma & \sigma^{-1}\gamma\gamma_1 \end{vmatrix} + \begin{vmatrix} \sigma^2-\sigma\delta+1 & \sigma^{-1}\gamma(\sigma^2-\sigma\delta+1) \\ \sigma^{-2}-\sigma^{-1}\delta+1 & \sigma\gamma(\sigma^{-2}-\sigma^{-1}\delta+1) \end{vmatrix}$$

$$= \gamma(\sigma-\sigma^{-1})\{\gamma\gamma_1+\sigma^{-2}(\sigma^2-\sigma\delta+1)^2\}.$$

Denoting, for the moment, the quantity in brackets by C, we observe that $C\sigma^2$ may be written

$$2(\sigma^2-\sigma\delta+1)\left(\sigma^2-\frac{a+\delta}{2}\sigma+1\right)$$

and hence does not vanish. If $\sigma-\sigma^{-1}=0$, then $\sigma^2-1=0$, contrary to hypothesis. The left member of the second abelian relation (2) is seen in like manner to be

$$\gamma(\sigma^{p^n}-\sigma^{-p^n})C^{p^n}.$$

If $\sigma^{p^{2n}}=\sigma$, it is only necesary that X_1 and X_2, Y_1 and Y_2 be conjugate with respect to the $GF[p^n]$. When we take μX_1 in place of X_1 and $\mu^{p^n}X_2$ in place of X_2, the canonical form (37) is preserved, as well as the abelian relations (3) and (4) just established. In case μ is the reciprocal of $\gamma(\sigma-\sigma^{-1})C$, the resulting transformation of indices satisfies also the abelian relations (2) and is therefore an abelian substitution in the $GF[p^{2n}]$ on two pairs of conjugate indices. In this case it follows that two substitutions S_2 having the same characteristic equation are conjugate within $SA(4, p^n)$.

If $\Delta(\kappa) = 0$ be irreducible, so that $\sigma^{p^{2n}} = \sigma^{-1}$, the replacement of X_i by μX_1 requires the replacement of Y_1 by $\mu^{p^{2n}} Y_1$, X_2 by $\mu^{p^n} X_2$, Y_2 by $\mu^{p^{3n}} Y_2$. The above expression $\gamma(\sigma - \sigma^{-1})C$ is then multiplied by $\mu^{p^{2n}+1}$, a mark of the $GF[p^{2n}]$; the product can not be made unity by choice of μ, since C belongs to the $GF[p^{2n}]$, while $\sigma - \sigma^{-1}$ does not. Hence S_2 is not reducible to the canonical form (37) by a special abelian substitution with conjugate indices. Nor is S_2 so reducible by a general abelian substitution; for the product

$$\mu^{p^{2n}+1}\gamma(\sigma - \sigma^{-1})C$$

differs from its (p^n)-th power; indeed, the product equals the negative of its (p^{2n})-th power. By a suitable choice of μ, $\mu^{p^{2n}+1}\gamma C = 1$, so that the left members of (2) become $\sigma - \sigma^{-1}$ and $\sigma^{p^n} - \sigma^{-p^n}$. Hence if T denote the transformation of indices reducing S_2 to the canonical form (37), then T replaces ϕ by ϕ', where

$$\phi \equiv \begin{vmatrix} \xi_1 & \eta_1 \\ \bar{\xi}_1 & \bar{\eta}_1 \end{vmatrix} + \begin{vmatrix} \xi_2 & \eta_2 \\ \bar{\xi}_2 & \bar{\eta}_2 \end{vmatrix}, \quad \phi' \equiv (\sigma - \sigma^{-1}) \begin{vmatrix} \xi_1 & \eta_1 \\ \bar{\xi}_1 & \bar{\eta}_1 \end{vmatrix} + (\sigma^{p^n} - \sigma^{-p^n}) \begin{vmatrix} \xi_2 & \eta_2 \\ \bar{\xi}_2 & \bar{\eta}_2 \end{vmatrix},$$

when T operates cogrediently upon the indices ξ_i, η_i and $\bar{\xi}_i$, $\bar{\eta}_i$. If T' denote the transformation of indices which reduces a second substitution S_2' to the canonical form (37), the product $A \equiv T'T^{-1}$ leaves ϕ absolutely invariant and transforms S_2' into S_2. In view of the conjugacy of the indices X_i, Y_i and the invariance of ϕ under A, the latter may be expressed as a special abelian substitution on ξ_i, η_i with coefficients in the $GF[p^n]$. Hence, *if two substitutions S_2 have the same characteristic determinant and if the latter be irreducible or the product of two irreducible quadratic factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, they are conjugate within the group $SA(4, p^n)$.*

§ 18.

Let S be an abelian substitution whose characteristic determinant is irreducible in the $GF[p^n]$, so that S may be reduced to the canonical form (37), where $\sigma^{p^{2n}+1} = 1$. Of the solutions of the latter, only $\sigma = \pm 1$ satisfy also $\sigma^{p^{2n}-1} = 1$, so that there remain $p^{2n} - 1$ or 2^{2n} suitable values of σ. Replacing σ by σ^{-1}, we obtain from (37) a substitution which is transformed into (37) by M_1M_2; replacing σ by σ^{-p^n}, we obtain the transform of (37) by $P_{12}M_1$. Replacing σ by σ^{p^n}, we obtain the transform of (37) by $P_{12}M_2$. Any new replacement of σ leads to a substitution not conjugate with (37). Hence there are $\frac{1}{4}(p^{2n} - 1)$ or $\frac{1}{4}2^{2n}$ non-conjugate types (37). An abelian substitution S_1 commutative with an abelian substitution S having the canonical form (37) has simultaneously the canonical form

$$X_1' = \rho X_1, \quad Y_1' = \rho^{-1} Y_1, \quad X_2' = \rho^{p^n} X_2, \quad Y_2' = \rho^{-p^n} Y_2 \quad (\rho^{p^{2n}+1} = 1),$$

so that there are $p^{2n} + 1$ such substitutions S_1. Hence S is conjugate with $p^{4n}(p^{2n} - 1)^2$ substitutions within $SA(4, p^n)$.

§ 19.

An abelian substitution S whose characteristic determinant is the product of two irreducible quadratic factors $\kappa^2 - \rho\kappa + \tau$, $\tau \neq 1$, is reducible to the canonical form (37), where $\sigma^{p^{2n}-1} = 1$, $\sigma^{p^n} \neq \sigma$, $\sigma^{p^n} \neq \sigma^{-1}$. The $(p^n-1)^2$ or $2^{2n} - 2 \cdot 2^n$ suitable values of σ give, in sets of four, conjugate substitutions. Hence there are $\frac{1}{4}(p^n-1)^2$ or $\frac{1}{4}2^n(2^n-2)$ non-conjugate types. Each is commutative with $p^{2n}-1$ substitutions and therefore conjugate with exactly $p^{4n}(p^{4n}-1)$ substitutions of $SA(4, p^n)$.

§ 20.

An abelian substitution S whose characteristic determinant is the product of two distinct irreducible factors of the form $\kappa^2 - A\kappa + 1$ is conjugate within $SA(4, p^n)$ with a substitution (30) and therefore is reducible to the canonical form

$$(38) \qquad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = \mu X_2, \quad Y_2' = \mu^{-1} Y_2,$$

where $\lambda^{p^n+1} = 1$, $\mu^{p^n+1} = 1$, $\lambda^{p^n-1} \neq 1$, $\mu^{p^n-1} \neq 1$, $\mu \neq \lambda$, $\mu \neq \lambda^{-1}$. Of the p^n+1 solutions of $\lambda^{p^n+1} = 1$, $\lambda = \pm 1$ are to be excluded; then of the p^n+1 solutions of $\mu^{p^n+1} = 1$, $\mu = \pm 1$ and $\mu = \lambda$, λ^{-1} are to be excluded. Hence there are $(p^n-1)(p^n-3)$ or $2^n(2^n-2)$ pairs of suitable values λ, μ. But λ and λ^{-1} are interchanged upon transforming (38) by M_1; μ and μ^{-1} upon transforming by M_2; λ and μ, λ^{-1} and μ^{-1} upon transforming by P_{12}. Hence eight of the pairs of values λ, μ lead to conjugate types, so that there are exactly $\frac{1}{8}(p^n-1)(p^n-3)$ or $\frac{1}{8}2^n(2^n-2)$ non-conjugate types (38). Each is commutative with exactly $(p^n+1)^2$ substitutions having the canonical form

$$X_1' = \tau^l X_1, \quad Y_1' = \tau^{-l} Y_1, \quad X_2' = \tau^m X_2, \quad Y_2' = \tau^{-m} Y_2,$$

τ being a primitive root of the equation $\tau^{p^n+1} = 1$ and l and m being arbitrary integers. Each type represents a set of $p^{4n}(p^{2n}+1)(p^n-1)^2$ conjugate substitutions of $SA(4, p^n)$.

§ 21.

An abelian substitution S whose characteristic determinant is the square of an irreducible quadratic is either conjugate within $SA(4, p^n)$ with a substitution (30) having $A' = A$ or else with a substitution S_2 satisfying (31). In the former case, S has the canonical form

$$(39) \qquad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = \lambda Y_2, \quad Y_2' = \lambda^{-1} Y_2,$$

where $\lambda^{p^n+1} = 1$, $\lambda \neq \pm 1$. There are $\frac{1}{2}(p^n-1)$ or 2^{n-1} types not conjugate within $SA(4, p^n)$. An abelian substitution S_1, commutative with S has simultaneously the canonical form

$$X_1' = aX_1 + \beta X_2, \qquad X_2' = \gamma X_1 + \delta X_2,$$
$$Y_1' = a^{p^n} Y_1 + \beta^{p^n} Y_2, \quad Y_2' = \gamma^{p^n} Y_1 + \delta^{p^n} Y_2,$$

subject to the abelian conditions *

$$a^{p^n+1} + \beta^{p^n+1} = 1, \quad \gamma^{p^n+1} + \delta^{p^n+1} = 1, \quad a\gamma^{p^n} + \beta\delta^{p^n} = 0.$$

But these are the conditions that $\begin{pmatrix} a & \beta \\ \gamma & \delta \end{pmatrix}$ shall be a binary hyperorthogonal substitution, the number of which is $(p^n+1)p^n(p^{2n}-1)$. Hence each canonical form (39) represents a set of $p^{3n}(p^{2n}+1)(p^n-1)$ conjugate substitutions of $SA(4, p^n)$.

§ 22.

A substitution S_2 satisfying (31) has the canonical form

(40) $X_1' = \sigma X_1, \quad Y_1' = \sigma^{-1} Y_1 + \sigma^{-1} X_2, \quad X_2' = \sigma^{-1} X_2, \quad Y_2' = \sigma Y_2 + \sigma X_1,$

σ being a root of $\Delta(\kappa) = 0$. As in § 17, we employ new indices

$$X_1 \equiv -\gamma\xi_1 + \sigma\gamma\gamma_1\eta_1 + (\sigma^2 - \sigma\delta + 1)\xi_2 + \sigma^{-1}\gamma(\sigma^2 - \sigma\delta + 1)\eta_2,$$

and X_2 obtained from X_1 by replacing σ by $\sigma^{p^n} \equiv \sigma^{-1}$; but for Y_1 and Y_2 we now take the functions

$$Y_1 \equiv \gamma\xi_1 + (\sigma^{-2} - 1)\xi_2 + \gamma(\delta - 2\sigma)\eta_2, \quad Y_2 \equiv \gamma\xi_1 + (\sigma^2 - 1)\xi_2 + \gamma(\delta - 2\sigma^{-1})\eta_2.$$

The determinant of the transformation of indices is seen to be

$$-\gamma^3\gamma_1(\sigma - \sigma^{-1})^4 \neq 0.$$

Upon replacing σ by σ^{-1} in (40), we obtain the transform of (40) by P_{12}. Hence there are $\frac{1}{2}(p^n - 1)$ or 2^{n-1} non-conjugate types (40). Each is commutative only with the substitutions

$$X_1' = aX_1, \quad Y_1' = a^{p^n} Y_1 + d^{p^n} X_2, \quad X_2' = a^{p^n} X_2, \quad Y_2' = aY_2 + dX_1,$$

where $a^{p^n+1} = 1$, $a^{p^n} d = ad^{p^n}$, so that $d = a\kappa$, κ a mark of the $GF[p^n]$, giving $p^n(p^n+1)$ commutative substitutions. Hence each type represents a set of $p^{3n}(p^{4n}-1)(p^n-1)$ conjugate substitutions within $SA(4, p^n)$.

§ 23. *Summary of the preceding results.*

The numerical results obtained in the preceding investigation are collected into the following table. The mark μ denotes 1 or a particular not-square ν when $p > 2$; while $\mu = 1$, if $p = 2$. Also, θ denotes $\frac{1}{2}$ or 1 according as $p > 2$ or $p = 2$. Finally, for $p > 2$, $\epsilon = \pm 1$ according as $p^n = 4l \pm 1$. By the "number of types" is meant the number of non-conjugate types of the speci-

* Indeed, S given by (30) with $A' = A$ may be reduced to its canonical form by an abelian substitution (not necessarily special), so that the canonical form of S_1 satisfies the conditions (2), (3), (4).

fied form within $SA(4, p^n)$. As a check upon the enumeration it was verified that the sum of the products of the number of conjugate substitutions of each type (fourth column) by the number of types (second or third column) gives the order $N \equiv p^{4n}(p^{4n}-1)(p^{2n}-1)$ of the group.

TABLE OF THE NON-CONJUGATE TYPES OF OPERATORS OF THE GROUP $SA(4, p^n)$ OF ORDER N.

Type.	Number of types.		Number of conjugate substitutions of each type.
	$p>2$	$p=2$	
(6)	$\frac{1}{8}(p^n-3)(p^n-5)$	$(2^{n-1}-1)(2^{n-2}-1)$	$p^{4n}(p^{2n}+1)(p^n+1)^2$
(7)	$\frac{1}{2}(p^n-3)$	$2^{n-1}-1$	$p^{3n}(p^{2n}+1)(p^n+1)$
(8)	$\frac{1}{2}(p^n-3)$	$2^{n-1}-1$	$p^{3n}(p^{4n}-1)(p^n+1)$
(9), $\beta=0$	p^n-3	$2^{n-1}-1$	$p^{3n}(p^{2n}+1)(p^n+1)$
(9), $\beta=\mu$	$2(p^n-3)$	$2^{n-1}-1$	$\theta p^{3n}(p^{4n}-1)(p^n+1)$
$T_{1\,\pm1}T_{2\,\pm1}$	2	1	1
$L_{1\,\mu}T_{1\,\pm1}T_{2\,\pm1}$	4	1	$\theta(p^{4n}-1)$
$L_{11}L_{21}T_{1\,\pm1}T_{2\,\pm1}$	2		$\frac{1}{2}p^n(p^{4n}-1)(p^n+\epsilon)$
		1	$(2^{4n}-1)(2^{2n}-1)$
$L_{1\,\nu}L_{21}T_{1\,\pm1}T_{2\,\pm1}$	2		$\frac{1}{2}p^n(p^{4n}-1)(p^n-\epsilon)$
(13)		1	$2^{4n}-1$
R_β, R_0		2	$\frac{1}{2}2^{2n}(2^{4n}-1)(2^{2n}-1)$
$A_\mu T_{1\,\pm1}T_{2\,\pm1}$	4		$\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$
(18)	$\frac{1}{4}(p^n-1)(p^n-3)$	$2^{n-1}(2^{n-1}-1)$	$p^{4n}(p^{4n}-1)$
(19), $b=0$	p^n-1	2^{n-1}	$p^{3n}(p^{2n}+1)(p^n-1)$
(19), $b=\mu$	$2(p^n-1)$	2^{n-1}	$\theta p^{3n}(p^{4n}-1)(p^n-1)$
(37), $\sigma^{p^{2n}+1}=1$	$\frac{1}{4}(p^{2n}-1)$	2^{2n-2}	$p^{4n}(p^{2n}-1)^2$
(37), $\sigma^{p^{2n}-1}=1$	$\frac{1}{4}(p^n-1)^2$	$2^{n-1}(2^{n-1}-1)$	$p^{4n}(p^{4n}-1)$
(38)	$\frac{1}{8}(p^n-1)(p^n-3)$	$2^{n-2}(2^{n-1}-1)$	$p^{4n}(p^{2n}+1)(p^n-1)^2$
(39)	$\frac{1}{2}(p^n-1)$	2^{n-1}	$p^{3n}(p^{2n}+1)(p^n-1)$
(40)	$\frac{1}{2}(p^n-1)$	2^{n-1}	$p^{3n}(p^{4n}-1)(p^n-1)$
$T_{1\,-1}$	1		$p^{2n}(p^{2n}+1)$
$L_{1\,\mu}T_{1\,-1}$ $L_{2\,\mu}T_{1\,-1}$	4		$\frac{1}{2}p^{2n}(p^{4n}-1)$
$L_{11}T_{1\,-1}L_{2\,\mu}$ $L_{1\,\nu}T_{1\,-1}L_{2\,\mu}$	4		$\frac{1}{4}p^{2n}(p^{4n}-1)(p^{2n}-1)$

§ 24. *The group $SA(4, 2)$ and the symmetric group on six letters.*

A second check upon the above results is furnished by a consideration of the case $p^n = 2$, when the group $SA(4, 2)$ is holoedrically isomorphic with the symmetric group on six letters.* The isomorphism may be established by the correspondence of generators:

$$(12) \sim M_1, \quad (23) \sim L_{11}, \quad (34) \sim S, \quad (45) \sim L_{21}, \quad (56) \sim M_2,$$

where

$$S \equiv \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad S' \equiv L_{11}L_{21}SL_{21}L_{11} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \sim (25).$$

Since M_i transforms L_{i1} into L'_{i1}, $L'_{11} \sim (13)$, $L'_{21} \sim (46)$. Then †

$$R_1 \equiv L_{21}M_2S'M_2L'_{21} \sim (2456), \quad R_0 \equiv M_2L'_{11}S'L_{21} \sim (13)(2456),$$

$$[13] \equiv L'_{11}L'_{21}S' \sim (13)(46)(25).$$

By § 12, there is a single type $L_{21}M_2$ given by (19′) for $b = 0$ and a single type $L'_{11}L_{21}M_2$ given by (19′) for $b = 1$, The single type (39) may be represented by $L_{11}M_1L_{21}M_2$. The single types (40) and (37), $\sigma^{p^{2n}+1} = 1$, may be represented respectively by

$$[40] \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \equiv M_1[13]M_2,$$

$$[37] \equiv \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \equiv L'_{21}M_1[13]M_2.$$

For $p^n = 2$ the above table gives the following types of abelian substitutions:

* Proceedings of the London Mathematical Society, vol. 31, p. 40, 1899.

† The abelian substitution numbered (13) is now referred to as [13].

Type.	Number of conjugates.	Corresponding substitution on six letters.
identity	1	identity
L_{11}	15	(23)
$L_{11}L_{21}$	45	(23)(45)
[13]	15	(13)(25)(46)
R_1	90	(2456)
R_0	90	(13)(2456)
$L_{21}M_2$	40	(465)
$L'_{11}L_{21}M_2$	120	(13)(465)
[37]	144	(16523)
$L_{11}M_1L_{21}M_2$	40	(123)(465)
[40]	120	(164523)

The third column gives every type of substitution on six letters and the number of conjugates to each type is given by the second column. In view of the independence of the two determinations of the types of substitutions of $SA(4, 2)$, the check is a complete one.

§ 25. *Operators of the simple group* $A(4, p^n)$, $p > 2$.

For $p = 2$, $n > 1$, the group $SA(4, p^n)$ is simple; for $p > 2$, it has the maximal invariant subgroup composed of the identity and $T \equiv T_{1-1}T_{2-1}$, the quotient-group $A(4, p^n)$ being simple.* In view of the importance of the latter group, we proceed to determine, by means of the earlier results, the distribution of its operators into complete sets of conjugate operators.

In the table (§ 23) of the non-conjugate types of substitutions within $SA(4, p^n)$, $p > 2$, the types are grouped into sets (each set being exhibited in a single line of the table, except the last two sets) such that types S and ST always belong to the same set. If S be not conjugate with ST within $SA(4, p^n)$, the number of conjugates with S within $SA(4, p^n)$ equals the number of conjugates with S within $A(4, p^n)$.† If, however, there exists a substitution V in $SA(4, p^n)$ which transforms S into ST, the number of conjugates with S in $A(4, p^n)$ equals one-half the number of conjugates with S in $SA(4, p^n)$; indeed, if W transforms S_1 into S, then WVW^{-1} will transform S_1 into S_1T.

A type S of the group $SA(4, p^n)$, $p > 2$, will be called special and denoted

*Quarterly Journal of Mathematics, vol. 29, pp. 169–178, 1897; vol. 31, pp. 383–4, 1899.

† In the quotient-group, S and ST become the same operator. It is convenient to denote the latter by S, the context sufficing to avoid confusion.

S_s if S be conjugate with ST within the group. For example, there is a single special type $(7)_s$,

$$\xi_1' = i\xi_1, \quad \eta_1' = -i\eta_1, \quad \xi_2' = -i\xi_2, \quad \eta_2' = i\eta_2 \qquad (i^2 = -1),$$

occurring if and only if -1 be a square in the $GF[p^n]$, viz., if $p^n = 4l+1$. In general, a type can be special only when the negative of each root of the characteristic equation is also a root.

For special types (6), three cases are to be examined. First, if $-\kappa = \kappa^{-1}$, so that $-\kappa^{-1} = \kappa$, then must $-\lambda = \lambda^{-1}$; hence $\kappa^2 = -1$, $\lambda^2 = -1$, so that $\lambda = \kappa$ or κ^{-1}, contrary to the hypothesis for type (6). Second, if $-\kappa = \lambda$, we have the special type

$$(6)_s \qquad \xi_1' = \kappa\xi_1, \quad \eta_1' = \kappa^{-1}\eta_1, \quad \xi_2' = -\kappa\xi_2, \quad \eta_2' = -\kappa^{-1}\eta_2,$$

which may be transformed into $(6)_s T$ by P_{12}. By the hypotheses for a type (6), $\kappa \neq 0$, $\kappa^2 \neq 1$, $\kappa^2 \neq -1$, the latter having solutions in the $GF[p^n]$ if, and only if, $p^n = 4l+1$. Hence there are $\frac{1}{4}(p^n - 5)$ non-conjugate special types $(6)_s$ if $p^n = 4l+1$ and $\frac{1}{4}(p^n - 3)$ such types if $p^n = 4l-1$. Third, if $-\kappa = \lambda^{-1}$, the resulting special type is transformed into $(6)_s$ by M_2.

There is no special type (18), since $a^2 = -1$, $\lambda^2 = -1$ require that a and λ belong to the same field, contrary to hypothesis.

To show that type (37), $\sigma^{p^{2n}+1} = 1$, is never special, we consider three cases. If $-\sigma = \sigma^{-1}$, then $\sigma^2 = -1$, while $p^{2n}+1$ is not divisible by 4. If $-\sigma = \sigma^{p^n}$, then $\sigma^{p^{2n}} = -\sigma^{p^n} = \sigma \neq \sigma^{-1}$. Similarly, for $-\sigma = \sigma^{-p^n}$, $\sigma^{p^{2n}} = -\sigma^{-p^n} = \sigma \neq \sigma^{-1}$.

To determine the special types (37), $\sigma^{p^{2n}-1} = 1$, $\sigma^{p^n-1} \neq 1$, $\sigma^{p^n+1} \neq 1$, we examine the three cases. If $-\sigma = \sigma^{-1}$, either $\sigma^{p^n-1} = 1$ or $\sigma^{p^n+1} = 1$, contrary to hypothesis. If $-\sigma = \sigma^{p^n}$, each solution of $\sigma^{p^n-1} = -1$, such that $\sigma^2 \neq -1$ and therefore $\sigma^{p^n+1} \neq 1$, furnishes a special type; there are $p^n - 1$ or $p^n - 3$ such values of σ according as $p^n = 4l \pm 1$. If $-\sigma = \sigma^{-p^n}$, each solution of $\sigma^{p^n+1} = -1$, such that $\sigma^2 \neq -1$ and therefore $\sigma^{p^n-1} \neq 1$, furnishes a special type; there are $p^n \mp 1$ such values of σ according as $p^n = 4l \pm 1$. The two sets of values for σ are wholly distinct since $\sigma^2 \neq 1$. Hence there are $2p^n - 2$ values σ, whatever be the form of p^n, $p > 2$. Hence there are $\frac{1}{2}(p^n - 1)$ special types (37) when $\sigma^{p^{2n}-1} = 1$.

Type (38) is not special for $-\lambda = \lambda^{-1}$, $-\mu = \mu^{-1}$, since then $\mu = \lambda$ or λ^{-1}, but is special for $\mu = -\lambda$, viz.:

$$(38)_s \quad X_1' = \lambda X_1, \quad Y_1' = \lambda^{-1} Y_1, \quad X_2' = -\lambda X_2, \quad Y_2' = -\lambda^{-1} Y_2,$$

since P_{12} transforms it into $(38)_s T$. The number of solutions of $\lambda^{p^n+1} = 1$, $\lambda^2 \neq 1$, $\lambda^2 \neq -1$, is $p^n - 1$ or $p^n - 3$, according as $p^n = 4l \pm 1$.

Of the remaining types in the table of § 23, it may be determined by inspection what special types, if any, exist. Our results may be combined in the following table:

TABLE OF THE NON-CONJUGATE TYPES*
OF OPERATORS OF THE SIMPLE GROUP $A(4, p^n)$, $p > 2$,
OF ORDER $\frac{1}{2}p^{4n}(p^{4n}-1)(p^{2n}-1)$.

Type.	Number of distinct types. $p^n = 4l+1$, $p^n = 4l-1$	Number of operators conjugate with each type.
$(6)_s$	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$\frac{1}{2}p^{4n}(p^{2n}+1)(p^n+1)^2$
(6)	$\frac{1}{16}(p^n-5)^2$, $\frac{1}{16}(p^n-3)(p^n-7)$	$p^{4n}(p^{2n}+1)(p^n+1)^2$
$(7)_s$	1, 0	$\frac{1}{2}p^{3n}(p^{2n}+1)(p^n+1)$
(7)	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n+1)$
$(8)_s$	1, 0	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n+1)$
(8)	$\frac{1}{4}(p^n-5)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{4n}-1)(p^n+1)$
(9), $\beta = 0$	$\frac{1}{2}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n+1)$
(9), $\beta = \mu$	p^n-3	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n+1)$
identity	1	1
$L_{1\mu}$	2	$\frac{1}{2}(p^{4n}-1)$
$L_{11}L_{21}$	1	$\frac{1}{2}p^n(p^{4n}-1)(p^n+\epsilon)$
$L_{1\nu}L_{21}$	1	$\frac{1}{2}p^n(p^{4n}-1)(p^n-\epsilon)$
A_μ	2	$\frac{1}{2}p^{2n}(p^{4n}-1)(p^{2n}-1)$
(18)	$\frac{1}{8}(p^n-1)(p^n-3)$	$p^{4n}(p^{4n}-1)$
(19), $b = 0$	$\frac{1}{2}(p^n-1)$	$p^{3n}(p^{2n}+1)(p^n-1)$
(19), $b = \mu$	p^n-1	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n-1)$
(37), $\sigma^{p^{2n}+1}=1$	$\frac{1}{8}(p^{2n}-1)$	$p^{4n}(p^{2n}-1)^2$
$(37)_s$, $\sigma^{p^{2n}-1}=1$	$\frac{1}{2}(p^n-1)$	$\frac{1}{2}p^{4n}(p^{4n}-1)$
(37), $\sigma^{p^{2n}-1}=1$	$\frac{1}{8}(p^n-1)(p^n-3)$	$p^{4n}(p^{4n}-1)$
$(38)_s$	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$\frac{1}{2}p^{4n}(p^{2n}+1)(p^n-1)^2$
(38)	$\frac{1}{16}(p^n-1)(p^n-5)$, $\frac{1}{16}(p^n-3)^2$	$p^{4n}(p^{2n}+1)(p^n-1)^2$
$(39)_s$	0, 1	$\frac{1}{2}p^{3n}(p^{2n}+1)(p^n-1)$
(39)	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{2n}+1)(p^n-1)$
$(40)_s$	0, 1	$\frac{1}{2}p^{3n}(p^{4n}-1)(p^n-1)$
(40)	$\frac{1}{4}(p^n-1)$, $\frac{1}{4}(p^n-3)$	$p^{3n}(p^{4n}-1)(p^n-1)$
T_{1-1}	1	$\frac{1}{2}p^{2n}(p^{2n}+1)$
$L_{1\mu}T_{1-1}$	2	$\frac{1}{2}p^{2n}(p^{4n}-1)$
$L_{11}L_{2\nu}T_{1-1}$	1	$\frac{1}{4}p^{2n}(p^{4n}-1)(p^{2n}-1)$
$L_{1\mu}L_{2\mu}T_{1-1}$	2	$\frac{1}{8}p^{2n}(p^{4n}-1)(p^{2n}-1)$

* A type marked s is special, otherwise a type is not special except for T_{1-1} and $L_{1\mu}L_{2\mu}T_{1-1}$.

In the table μ denotes 1 or a particular not-square ν; ϵ denotes ± 1 according as $p^n = 4l \pm 1$. The total number of non-conjugate types in the simple group $A(4, p^n)$, $p > 2$, is the same function of p^n in the two cases $p^n = 4l \pm 1$, viz.:

$$\tfrac{1}{2}(p^{2n} + 1) + 3p^n + 6.$$

§ 26. *The operators of the simple group* $A(4, 3)$ *of order* 25920.

As proved by JORDAN, $SA(4, 3)$ is the group of the equation for the trisection of the periods of a hyperelliptic function of four periods. Moreover, the group of the equation for the determination of the 27 lines on a general cubic surface of the third order is of the same order 51840 as $SA(4, 3)$. After a certain square root has been adjoined to the realm of rationality, the group reduces to the quotient-group $A(4, 3)$ of order 25920. Hence the above two problems are essentially the same. In view of the importance of the group $A(4, 3)$, it is desirable to know the distribution of its operators into complete sets of conjugates and likewise for its cyclic subgroups. By § 25, there are exactly twenty types of non-conjugate operators in the group. It is desirable to have simple representatives in the group for each type. Type (39) may be represented by $M_1 M_2$; type (19), $b = 0$, by M_2; type (19), $b = 1$, by $M_2 L_{11}$; type (19), $b = -1$, by $M_2 L_{1-1}$; type (37), $\sigma^4 = -1$, by $P_{12} M_1$; type (40) by $P_{12} L_{1-1} T_{1-1}$; type (37), $\sigma^{10} = 1$, by S_2 for $\gamma_1 = 1$, $\gamma = a = \delta = -1$, viz.,

$$K \equiv \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix},$$

of characteristic determinant $\kappa^4 - \kappa^3 + \kappa^2 - \kappa + 1$. Of the preceding statements, the only ones requiring further proof are those concerning the representatives $P_{12} M_1$ and $P_{12} L_{1-1} T_{1-1}$; but the former has the characteristic determinant $(\kappa^2 + \kappa - 1)(\kappa^2 - \kappa - 1)$, each factor being irreducible modulo 3, and hence has the canonical form (37), $\sigma^8 = 1$; while the latter is reduced to the canonical form (40) by the abelian transformation

$$X_1 = \eta_1 + \sigma\eta_2, \quad Y_1 = \xi_1 - \sigma\xi_2 - \sigma\eta_2, \quad X_2 = \eta_1 - \sigma\eta_2, \quad Y_2 = \xi_1 + \sigma\xi_2 + \sigma\eta_2,$$

where $\sigma^2 \equiv -1 \pmod 3$. We have therefore, by § 25, the complete list of types of operators of $A(4, 3)$ together with their periods and the number of their conjugates within $A(4, 3)$, as given on the following page.

§ 27. *Cyclic subgroups of the simple group* $A(4, 3)$.

To determine the distinct types of cyclic subgroups of $A(4, 3)$, it is necessary to find what powers of each type of substitution are conjugate with

Type.	Period.	Conjugates.	Type.	Period.	Conjugates.
L_{11}	3	40	identity	1	1
L_{1-1}	3	40	A_1	9	2880
$L_{11}L_{21}$	3	240	A_{-1}	9	2880
$L_{1-1}L_{21}$	3	480	M_2	4	540
T_{1-1}	2	45	M_2L_{11}	12	2160
$L_{11}T_{1-1}$	6	360	M_2L_{1-1}	12	2160
$L_{1-1}T_{1-1}$	6	360	K	5	5184
$L_{11}L_{2-1}T_{1-1}$	6	1440	$P_{12}M_1$	4	3240
$L_{11}L_{21}T_{1-1}$	6	720	M_1M_2	2	270
$L_{1-1}L_{2-1}T_{1-1}$	6	720	$P_{12}L_{1-1}T_{1-1}$	6	2160

that type. Thus, L_{11} is not conjugate with its square L_{1-1}; $L_{11}L_{21}$ is conjugate with its square. A_{-1} is transformed into A_1^{-1} by $T_{2-1}L'_{21}$. Since $A_1^3 = L'_{1-1}$ and $A_1^6 = L'_{11}$ are not conjugate, A_1 is not conjugate with either A_1^2 or A_1^5, so that the latter are conjugate with A_1^{-1}. Hence their squares A_1^4, A_1 and A_1^7 are conjugate. Hence A_1 generates a cyclic group self-conjugate only under a G_{27}. Again, M_2 is transformed into M_2^3 by the abelian substitution (modulo 3) $\xi'_2 = \xi_2 + \eta_2$, $\eta'_2 = \xi_2 - \eta_2$. Hence M_2L_{11} is conjugate with its seventh power $M_2^3L_{11}$, so that the fifth and eleventh powers are conjugate. The latter is $M_2^3L_{1-1}$ and is consequently conjugate with M_2L_{1-1} and hence (by the above table) not conjugate with M_2L_{11} itself. Hence M_2L_{11} is conjugate only with one other generator of the same cyclic group. The fact that $P_{12}L_{1-1}T_{1-1}$ is conjugate with its reciprocal within $A(4, 3)$ may be simply verified by observing that the canonical form (40) is transformed into its reciprocal by the abelian substitution $P_{12}T_{2-1}$ on the indices X_i, Y_i. In a similar way, $P_{12}M_1$ is shown to be conjugate with its reciprocal. Finally, there being but a single type of substitutions of period 5, K must be conjugate with K^2, K^3, K^4. We have therefore the following complete list of the distinct types of cyclic subgroups of $A(4, 3)$, together with the number of conjugates to each cyclic group.

Type of generator.	Conjugate cyclic groups.	Generator.	Groups.
L_{11}	40	A_1	960
$L_{11}L_{21}$	120	M_2	270
$L_{1-1}L_{21}$	240	M_2L_{11}	1080
T_{1-1}	45	K	1296
$L_{11}T_{1-1}$	360	$P_{12}M_1$	1620
$L_{11}L_{2-1}T_{1-1}$	720	M_1M_2	270
$L_{11}L_{21}T_{1-1}$	720	$P_{12}L_{1-1}T_{1-1}$	1080

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CERTAIN CASES IN WHICH THE VANISHING OF THE WRONSKIAN IS A SUFFICIENT CONDITION FOR LINEAR DEPENDENCE*

BY

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PEANO in Mathesis, vol. 9 (1889), p. 75 and p. 110 seems to have been the first to point out that the identical vanishing of the Wronskian of n functions of a single variable is not in all cases a sufficient condition for the linear dependence of these functions.† At the same time he indicated a case in which it *is* a sufficient condition,‡ and suggested the importance of finding other cases of the same sort. Without at first knowing of PEANO'S work, I was recently led to this same question, and found a case not included in PEANO'S in which the identical vanishing of the Wronskian is a sufficient condition.§ It is my purpose in the present paper to consider these cases and others of a similar nature.

By far the most important case in which the identical vanishing of the Wronskian is a sufficient condition for linear dependence is that in which the functions in question are at every point of a certain region analytic functions, whether of a real or complex variable is, of course, immaterial. This case requires no further treatment here.

We shall therefore be concerned exclusively with the case in which the independent variable x is real. This variable we will suppose to be confined to an interval I which may be finite or infinite, and if limited in one or both directions may or may not contain the end points. In some of the proofs we shall use a subinterval $a \leqq x \leqq b$ of I;‖ this subinterval we call I'.

Whether the functions are real or complex is immaterial.

We use the symbol $\equiv$ to denote an identity, i. e., an equality which holds at every point of the interval we are considering.

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† It is of course a necessary condition provided the functions have finite derivatives of the first $n-1$ orders at every point of the region in question.

‡ See § 4 of the present paper.

§ See Bulletin of the American Mathematical Society, December, 1900, p. 120, and Annals of Mathematics, second series, vol. 2, p. 93.

‖ We suppose here that a and b are finite quantities.

§ 1. *The Fundamental Theorem.*

We consider first the special case of two functions.

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be functions of x which at every point of I have finite first derivatives, while u_1 does not vanish in I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0, \tag{1}$$

u_1 and u_2 are linearly dependent throughout I, and in particular

$$u_2 \equiv c u_1. \tag{2}$$

For dividing (1) by u_1^2 we have:

$$\frac{d}{dx}\left(\frac{u_2}{u_1}\right) \equiv 0.$$

Therefore:

$$\frac{u_2}{u_1} \equiv c.$$

We pass now to the general case which includes the case just considered.

THEOREM II. *Let $u_1(x), u_2(x), \cdots, u_n(x)$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish in I; then if the Wronskian W of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent throughout I, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

In the Wronskian:

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix},$$

we denote by $W_1, W_2, \cdots, W_n$ the minors corresponding to the elements of the last row. We have then:

$$W_1 u_1^{(i)} + W_2 u_2^{(i)} + \cdots + W_n u_n^{(i)} \equiv 0 \quad (i = 0, 1, \cdots, n-1).$$

Differentiating each of the first $n-1$ of these identities and subtracting from it the one next following we get:

$$W_1' u_1^{(i)} + W_2' u_2^{(i)} + \cdots + W_n' u_n^{(i)} \equiv 0 \quad (i = 0, 1, \cdots, n-2).$$

Let us add these identities together after having multiplied the i-th of them

$(i = 1, 2, \cdots, n-1)$ by the first minor of W_n corresponding to $u_1^{(i-1)}$. This gives:

$$W_1' W_n - W_n' W_1 \equiv 0 .$$

Now since by hypothesis W_n does not vanish in I we have by theorem I:

$$W_1 \equiv - c_1 W_n .$$

In the same way:

$$\begin{aligned} W_2 &\equiv - c_2 W_n , \\ &\cdots \\ W_{n-1} &\equiv - c_{n-1} W_n . \end{aligned}$$

Therefore the identity

$$W_1 u_1 + W_2 u_2 + \cdots + W_n u_n \equiv 0 ,$$

can be written:

$$W_n(- c_1 u_1 - c_2 u_2 \cdots c_{n-1} u_{n-1} + u_n) \equiv 0 ,$$

and, since W_n does not vanish, our theorem follows at once.*

§ 2. *A Generalization for the Case of Two Functions.*

THEOREM III.† *Let u_1 and u_2 be functions of x which at every point of I have finite derivatives of the first k orders $(k \geqq 1)$, while u_1, u_1', u_1'', $\cdots$, $u_1^{(k)}$ do not all vanish at any one point of I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0 ,$$

u_1 and u_2 are linearly dependent, and in particular:

$$u_2 \equiv c u_1 .$$

This theorem will evidently be established if we can prove it for every finite and perfect subinterval I' of I. We will therefore in our proof consider only the interval I'.

There cannot be more than a finite number of points in I' where $u_1 = 0$. For if there were these points would have at least one limiting point x_0 in I', and since u_1 is continuous it would vanish at x_0. By ROLLE'S theorem there would also be an infinite number of points where $u_1' = 0$ and these points would have x_0 as limiting point, and owing to the continuity of u_1' we should have $u_1'(x_0) = 0$. Proceeding in the same way we see that u_1'', u_1''', $\cdots$, $u_1^{(k-1)}$ would all vanish at x_0. That $u_1^{(k)}$ would also vanish at x_0 must be shown in a slightly

* This proof is merely a slight modification of the one given by FROBENIUS, Crelle, vol. 76 (1873), p. 238. Cf. also HEFFTER, *Lineare Differentialgleichungen*, p. 233.

† The special case $k = 1$ of this theorem was given by PEANO, l. c. Cf. also Annals of Mathematics, second series, vol. 2, p. 92.

different manner since we do not know that $u_1^{(k)}$ is continuous. This follows at once, however, from the fact that $u_1^{(k-1)}$ would vanish in every neighborhood of x_0. We thus see that if u_1 vanished at an infinite number of points in I' there would be a point x_0 where $u_1, u_1', \cdots, u_1^{(k)}$ all vanish, and this is contrary to hypothesis.

The points at which $u_1 = 0$ therefore divide the interval I' into a finite number of pieces throughout each of which theorem I tells that u_2 is a constant multiple of u_1, and owing to the continuity of u_1 and u_2 this relation must also hold at the extremities of the piece in question. It remains to show that this constant is the same for all the pieces. It will evidently be sufficient to consider two adjacent pieces separated by the point p. Suppose that in the piece to the left of p we have

$$u_2 = c_1 u_1,$$

and in the piece to the right,

$$u_2 = c_2 u_1.$$

Since the derivatives of u_1 and u_2 at p may be found either by differentiating to the right or to the left we have:

$$u_2^{(i)}(p) = c_1 u_1^{(i)}(p), \qquad (i = 1, 2, \cdots, k).$$

$$u_2^{(i)}(p) = c_2 u_1^{(i)}(p),$$

Therefore

$$(c_1 - c_2) u_1^{(i)}(p) = 0 \qquad (i = 1, 2, \cdots, k).$$

Now, since $u_1(p) = 0$, there must be at least one of the derivatives $u_1', u_1'', \cdots, u_1^{(k)}$ which does not vanish at p. Therefore

$$c_1 = c_2,$$

and our theorem is proved.

§ 3. *Two Extensions to the case of n Functions.*

THEOREM IV.* *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n - 2 + k$ orders ($k \geqq 1$), while the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ and its first k derivatives do not all vanish at any one point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ is identically zero $u_1, u_2, \cdots, u_n$ are linearly dependent, and in particular:*

$$u_n \equiv c_1 u_1 + c_2 u_2 + \cdots + c_{n-1} u_{n-1}.$$

The proof of this theorem is, in the main, the same as that of theorem II. We will therefore only point out the two points of difference.

1. We must use theorem III instead of theorem I to establish the relation:

* The special case $k = 1$ of this theorem was given by the writer, l. c.

$$W_1 \equiv -c_1 W_n .$$

2. From the identity:

$$W_n(-c_1u_1 - c_2u_2 \cdots c_{n-1}u_{n-1} + u_n) \equiv 0 ,$$

we can now infer only that *at the points where* $W_n \neq 0$,

$$u_n = c_1u_1 + \cdots + c_{n-1}u_{n-1} .$$

In order to prove that this equation also holds at the points where $W_n = 0$, we notice first that these points in any finite and perfect subinterval I' of I are finite in number as otherwise there would be (cf. the proof of theorem III) a point of I' where W_n, W_n', $\cdots$, $W_n^{(k)}$ all vanish. All the points where W_n vanishes are therefore isolated, and since the equation

$$u_n = c_1u_1 + \cdots + c_{n-1}u_{n-1}$$

holds everywhere except at these points it must on account of the continuity of the u's hold at these points also. Thus our theorem is proved.

A little reflection on the results so far obtained will suggest the question whether the theorem of the last section might not be extended to the case of n functions by requiring, not as we have just done, that W_n, W_n', $\cdots$, $W_n^{(k)}$, do not all vanish at any point of I, but that u_1 and a certain number of its derivatives shall not all vanish at any point of I. The following example shows, however, not only that the theorem thus suggested is not true, but that even when *no one of the u's vanishes at any point of* I the identical vanishing of the Wronskian is not necessarily a sufficient condition for linear dependence when we have more than two functions.

EXAMPLE. Consider the three functions:

$$u_1 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x \neq 0), \\ 1 & (x = 0), \end{cases} \qquad u_2 = \begin{cases} 1 + e^{-\frac{1}{x^2}} & (x > 0), \\ 1 & (x = 0), \\ 1 - e^{-\frac{1}{x^2}} & (x < 0), \end{cases} \qquad u_3 = 1 .$$

These three functions are obviously linearly independent in any interval including both positive and negative values of x. Moreover no one of them vanishes for any real value of x. Yet the Wronskian of u_1, u_2, u_3 is identically zero.

The following theorems V and VI, which run somewhat along the lines just indicated, are, however, true:

THEOREM V. *Let* u_1, u_2, $\cdots$, u_n *be functions of* x *which at every point of* I *have finite derivatives of the first* $n-1$ *orders, while no function* (*other than zero*) *of the form:*

$$g_1u_1 + g_2u_2 + \cdots + g_nu_n$$

(the g's being constants) vanishes together with its first $n-1$ derivatives at any point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes at any point p of I these functions are linearly dependent.

From the fact that the Wronskian vanishes at p follows the existence of n constants $c_1, c_2 \cdots, c_n$ not all zero and such that

$$c_1 u_1^{(i)}(p) + c_2 u_2^{(i)}(p) + \cdots + c_n u_n^{(i)}(p) = 0 \quad (i = 0, 1, \cdots, n-1),$$

i. e., the function $c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ vanishes together with its first $n-1$ derivatives at the point p, and must therefore be identically zero. Thus our theorem is proved.

THEOREM VI. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first k orders $(k > n-1)$, while no function (other than zero) of the form*

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n,$$

(the g's being constants) vanishes together with its first k derivatives at any point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically these functions are linearly dependent.

We prove this theorem first on the supposition that the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish identically.* In this case there exists a point p of I where the Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. Since this last named Wronskian is continuous it is different from zero throughout the neighborhood of p. We see then by applying II that there exist n constants $c_1, c_2, \cdots, c_n$ not all zero and such that the function

$$c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

is zero throughout the neighborhood of p. Accordingly this function vanishes together with its first k derivatives at p, and therefore vanishes identically. Thus our theorem is proved in this special case.

In order to prove the theorem in general we first notice that if $u_1 \equiv 0$ the u's are surely linearly dependent. If u_1 is not identically zero, consider in succession the Wronskians of u_1, u_2, of u_1, u_2, u_3, of u_1, u_2, u_3, u_4, etc. Suppose the first of these which vanishes identically is the Wronskian of $u_1, u_2, \cdots, u_m$ $(m \leqq n-1)$. Then since the Wronskian of $u_1, u_2, \cdots, u_{m-1}$ does not vanish identically, the special case of our theorem which we have already proved shows that $u_1, u_2, \cdots, u_m$ are linearly dependent. Accordingly $u_1, u_2, \cdots, u_n$ are linearly dependent, and our theorem is proved.

Theorems V and VI admit of immediate application to the theory of linear differential equations, as the following theorem shows.

* The proof of this part of the theorem has been modified since the paper was presented to the Society by making it depend on II instead of on the lemmas of § 5.

THEOREM VII. *Let $p_1, p_2, \cdots, p_n$ be functions of x which at every point of I are continuous, and let $y_1, y_2, \cdots, y_k (k \leqq n)$ be functions of x which at every point of I satisfy the differential equation:*

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0;$$

then the identical vanishing of the Wronskian of $y_1, y_2, \cdots, y_k$ (or in the case $k = n$ the vanishing of this Wronskian at a single point of I) is a sufficient condition for the linear dependence of $y_1, y_2, \cdots, y_k$.

This theorem follows at once from theorems V and VI when we recall the fact that a solution of the above written differential equation which vanishes together with its first $n-1$ derivatives at a point of I is necessarily identically zero.

§ 4. *Discussion of Peano's Theorems.* *

One of PEANO's results, as has already been stated, is the special case $k = 1$ of theorem III. Apart from this PEANO's results cover no case which is not also covered by the fundamental theorem of § 1. I propose to show this in the present section.

For this purpose we first establish the following:

LEMMA. *Let u_1 and u_2 be functions of x which at every point of I have finite first derivatives, while*

$$u_1 u_2' - u_2 u_1' \equiv 0;$$

if a point p exists in I at which $u_2 = 0$, while in every neighborhood of p lie points where $u_2 \neq 0$, then $u_1(p) = 0$.

For if $u_1(p) \neq 0$ we could, on account of the continuity of u_1, mark off a neighborhood of p throughout which u_1 does not vanish, and throughout which therefore by theorem I

$$u_2 = c u_1.$$

Since at p $u_1 \neq 0$ and $u_2 = 0$ we must have $c = 0$, but this would make u_2 vanish throughout the neighborhood of p, and this is contrary to hypothesis.

PEANO deduces the following theorem in the case of two functions. This theorem includes as a special case the theorem to which theorem III reduces when $k = 1$, and appears at first sight to go beyond it.

PEANO'S FIRST THEOREM. *Let u_1 and u_2 be functions of x which at every point of I have finite first derivatives, while u_1, u_2, u_1', u_2' do not all vanish at any point of I; then if*

$$u_1 u_2' - u_2 u_1' \equiv 0,$$

u_1 and u_2 are linearly dependent.

* See, besides the notes in Mathesis referred to at the beginning of this article, a paper by PEANO: Rendiconti della Accademia dei Lincei, ser. 5, vol. 6, 1° sem. (1897), p. 413.

The truth of this theorem will be established, and at the same time it will be proved that it covers no case which is not also covered by the special case $k=1$ of theorem III, if we can show that either there is no point of I where u_1 and u_1' both vanish, or there is no point of I where u_2 and u_2' both vanish. Assume then that there is a point where u_2 and u_2' both vanish. Here we distinguish between two cases:

(*a*) $u_2 \equiv 0$. Here $u_2 = u_2' = 0$ at every point of I, and therefore there can be no point in I where $u_1 = u_1' = 0$.

(*b*) u_2 is not identically zero. Then there exists a point p in I at which $u_2 = u_2' = 0$, but in whose every neighborhood lie points where $u_2 \neq 0$. Therefore by the above lemma $u_1(p) = 0$. We must therefore have $u_1'(p) \neq 0$. Accordingly there exists an ϵ such that throughout the interval $p < x < p + \epsilon$, and also throughout the interval $p > x > p - \epsilon$, u_1 does not vanish. Let us choose that one of these intervals in which lie points where $u_2 \neq 0$. By theorem I we have at every point of this interval, and therefore on account of the continuity of u_1 and u_2 also at p,

$$u_2 = cu_1,$$

where $c \neq 0$ as otherwise u_2 would vanish at every point of this interval. From this last equation we infer that

$$u_2'(p) = cu_1'(p).$$

Therefore since $u_2'(p) = 0$ and $c \neq 0$ we get $u_1'(p) = 0$. We are thus led to a contradiction, and therefore the case (*b*) cannot occur.

PEANO'S SECOND THEOREM. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first $n-1$ orders, while the Wronskians of these functions taken $n-1$ at a time do not all vanish at any point of I; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically $u_1, u_2, \cdots, u_n$ are linearly dependent.*

We will establish this theorem, and at the same time show that it covers no case which is not also covered by the fundamental theorem II, by proving that there must be one of the Wronskians $W_1, W_2, \cdots, W_n$ (to use the notation employed in the proof of theorem II) which does not vanish at any point of the interval I. Suppose each of these W's vanished in I. They cannot all vanish identically. Suppose that W_n is one of those which does not vanish identically. Then there exists a point p at which $W_n = 0$ but in whose every neighborhood lie points where $W_n \neq 0$.

Now by the reasoning used in the proof of theorem II we see that:

$$W_i' W_n - W_i W_n' \equiv 0 \qquad (i = 1, 2, \cdots, n-1).$$

Therefore, by our lemma, W_i vanishes at p $(i = 1, 2, \cdots, n-1)$ and this is contrary to hypothesis since W_n also vanishes at p.

§ 5. *A Theorem concerning Wronskians.*

I have now completed what I have to say on the subject of linear dependence. There remains however a theorem concerning Wronskians which I have found useful in the course of my work, although in the form which I have finally given to this paper no use has been made of it.

Before stating this theorem we will first establish two lemmas which we shall use in its proof.

Consider a matrix M of $n+m$ rows and n columns. Denote by D_i the n-rowed determinant obtained from M by striking out all of its $m+1$ last rows except the $(n-1+i)$-th row. Denote by M' the matrix obtained from M by striking out its last $m+1$ rows. Denote by Δ_i the $(n-1)$-rowed determinant obtained from M' by striking out its i-th column.

LEMMA I. *If* $D_1 = D_2 = \cdots = D_{m+1} = 0$, *and if* $\Delta_1, \Delta_2, \cdots, \Delta_n$ *are not all zero, then all the* n*-rowed determinants of* M *are zero.*

For denoting the element of M which stands in the i-th row and j-th column by a_{ij}, we have:

$$a_{i1}\Delta_1 - a_{i2}\Delta_2 + \cdots + (-1)^{n-1}a_{in}\Delta_n = 0 \qquad (i = 1, 2, \cdots, n+m),$$

and these form a set of $n+m$ homogeneous linear equations satisfied by the n Δ's which by hypothesis are not all zero.

LEMMA II. *Let* $u_1, u_2, \cdots, u_n$ *be functions of* x *which at every point of* I *have finite derivatives of the first* k *orders* $(k \geqq n)$, *while their Wronskian vanishes identically; then, except at points where the Wronskian of* $u_1, u_2, \cdots, u_{n-1}$ *is zero, all the* n*-rowed determinants of the matrix:*

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{vmatrix}$$

are zero.

We first prove this lemma in the case $k = n$. Here the determinant obtained from the above matrix by striking out the next to the last row is simply the derivative of the Wronskian of $u_1, u_2, \cdots, u_n$, and therefore also vanishes identically. The truth of our lemma thus follows at once from lemma I.

In order to prove the lemma in the general case we use the method of mathematical induction, and assume that the lemma has been proved when $k = k_1 - 1$. We wish to prove that the lemma also holds when $k = k_1$. Let us denote by M the above matrix when k has the value k_1, and by N the matrix obtained from M by striking out its last row; and let p be any point of I where the

Wronskian of $u_1, u_2, \cdots, u_{n-1}$ does not vanish. If then we can prove that the determinant:

$$D = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-2)} & u_2^{(n-2)} & \cdots & u_n^{(n-2)} \\ u_1^{(k_1)} & u_2^{(k_1)} & \cdots & u_n^{(k_1)} \end{vmatrix},$$

vanishes at p, it will follow at once from lemma I that all the n-rowed determinants of M vanish at p, since this is true of all the n-rowed determinants of N. In order to prove that D vanishes at p let us consider the $(k_1 - n + 1)$-th derivative of the Wronskian of $u_1, u_2, \cdots, u_n$. This derivative will of course vanish identically. If we compute its value we find that it consists of the sum of a number of n-rowed determinants of which D is one while the others are all determinants of the matrix N, and therefore vanish at p. Thus we see that D vanishes at p, and our lemma is proved.

THEOREM VIII. *Let $u_1, u_2, \cdots, u_{n+1}$ be functions of x which at every point of I have continuous derivatives of the first n orders; then if the Wronskian of $u_1, u_2, \cdots, u_n$ vanishes identically the Wronskian of $u_1, u_2, \cdots, u_{n+1}$ will vanish identically.*

Denote by M the matrix obtained from the Wronskian

$$W = \begin{vmatrix} u_1 & u_2 & \cdots & u_{n+1} \\ u_1' & u_2' & \cdots & u_{n+1}' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n)} & u_2^{(n)} & \cdots & u_{n+1}^{(n)} \end{vmatrix}$$

by striking out the last column. Then lemma II tells us that all the n-rowed determinants of M vanish except at the points where the Wronskians $\Delta_1, \Delta_2, \cdots, \Delta_n$ of the functions $u_1, u_2, \cdots, u_n$ taken $n-1$ at a time all vanish. Accordingly $W = 0$ except at these points. Let p be any such point of I. Our theorem will be proved if we can show that W vanishes at p.

We must distinguish two cases:

(*a*) $\Delta_1, \Delta_2, \cdots, \Delta_n$ do not all vanish identically throughout the neighborhood of p. There are therefore points in every neighborhood of p where the Δ's are not all zero, and where therefore $W = 0$; accordingly W must also vanish at p since it is a continuous function of x.*

(*b*) The Δ's all vanish identically throughout the neighborhood of p. Before

* This is the only point in the proof where use is made of the assumption that the nth derivatives of the u's are continuous. Would not the theorem still be true without this assumption?

proving in general that $W = 0$ for points of class (b) we will prove it in the simple case $n = 2$. Here we have two Δ's: $\Delta_1 = u_2$, $\Delta_2 = u_1$. Since these vanish identically in the neighborhood of p all the elements of the first two columns of W vanish at p, and therefore W vanishes at p.

We will now complete our proof by the method of mathematical induction by assuming that the theorem has been proved when we have less than $n + 1$ functions. Since each of the Δ's is the Wronskian of $n - 1$ of the functions $u_1, u_2, \cdots, u_n$ it follows that throughout the neighborhood of p the Wronskian of any n of the $n + 1$ functions $u_1, u_2, \cdots, u_{n+1}$ must vanish. Accordingly W also vanishes at p, as we see by expanding it according to the elements of its last row.

RAPALLO, ITALY, *December* 9, 1900.

AN ELEMENTARY PROOF OF A THEOREM OF STURM*

BY

MAXIME BÔCHER

We shall have to deal with two real solutions y_1 and y_2 of the differential equation:

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are throughout an interval $a \leqq x \leqq b$ real and continuous functions of the real variable x.† One of the most important of STURM'S results (Liouville's Journal, vol. 1 (1836), p. 106) is that, if y_1 and y_2 are linearly independent, between two successive roots of one lies one and only one root of the other.

The following generalization is (implicitly at least) contained in STURM'S paper, and from it what I have called STURM'S theorems of comparison for a single equation ‡ follow at once. It is my object in the present note to prove this theorem by a simple and elementary method which makes use only of a single property of y_1 and y_2, namely that a necessary and sufficient condition for their linear dependence is that $y_1y_2' - y_2y_1'$ should vanish at some point of the interval ab.§

The theorem in question may be stated as follows, and when it is so stated the method of proof is at once suggested:

Suppose that y_1 vanishes neither at a nor at b, and that y_2, if it does not vanish at a, satisfies the relation:

$$\frac{y_2'(a)}{y_2(a)} > \frac{y_1'(a)}{y_1(a)},$$

and, if it does not vanish at b, satisfies the relation:

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* Presented to the Society February 23, 1901. Received for publication January 14, 1901.

† We may, if we wish, allow p and q to have a finite number of discontinuities in this interval, provided that in the neighborhood of each discontinuity p and q remain finite, or, if they become infinite, do so in such a way that $|p|$ and $|q|$ can be integrated up to these points. See Bulletin of the American Mathematical Society, March, 1899, p. 276.

‡ Bulletin of the American Mathematical Society, May, 1898, pp. 366–367.

§ In a slightly different form I applied this method four years ago to the simple case stated above. See Bulletin of the American Mathematical Society, March, 1897, p. 210.

$$\frac{y_2'(b)}{y_2(b)} < \frac{y_1'(b)}{y_1(b)},$$

*and finally that y_2 does not vanish in the interval $a < x < b$; then y_1 vanishes once and only once in this interval.**

We first prove that y_1 must vanish at least once. For if it did not the function:

$$f(x) = \frac{y_2'(x)}{y_2(x)} - \frac{y_1'(x)}{y_1(x)}$$

would be continuous throughout the interval $a < x < b$. Moreover if $y_2(a) \neq 0$ we have $f(a) > 0$, if $y_2(a) = 0$ we have $f(a) = +\infty$. In any case therefore $f(x)$ is positive in the neighborhood of a. In the same way we see that $f(x)$ is negative in the neighborhood of b. Accordingly $f(x)$ must vanish at some point between a and b. Therefore at this point:

$$y_1 y_2' - y_2 y_1' = 0,$$

and hence y_1 and y_2 are linearly dependent. This however is clearly not the case.

If now y_1 vanished more than once in the interval $a < x < b$, let $x = a$ and $x = \beta$ be two successive points in this interval where $y_1 = 0$. Then, interchanging y_1 and y_2, and applying the part of the theorem we have proved to the interval $a < x < \beta$, we see that y_2 vanishes at least once between a and β, and this is contrary to hypothesis.

RAPALLO, ITALY, *December* 31, 1900.

* If $y_2(a) = y_2(b) = 0$ this theorem reduces to the simple theorem stated above.

ON THE DETERMINATION OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY ALGEBRAIC CURVES*

BY

HENRY FREEMAN STECKER

Introduction.

BELTRAMI has shown † that surfaces of constant curvature can be built upon the plane in such a manner that the geodetic lines shall go over into straight lines, and that this result is true for no other surface. He considered this as the simplest case of building one surface upon another in such a manner that the geodetic lines of one surface shall go over into the geodetic lines of the other. The general question was later solved by DINI. ‡ It is an immediate consequence of BELTRAMI'S memoirs that surfaces of constant curvature are the only surfaces that can be built conformally upon the plane in such a manner that the geodetic lines shall go over into straight lines or arcs of circles. § This latter fact suggests a generalization of BELTRAMI'S problem different from the one which he had in mind, i. e., to so build a surface conformally upon the plane in such a manner that the geodetic lines shall go over into algebraic curves. It is proposed to consider that question.

§ 1.

We consider a doubly infinite system of algebraic curves in the plane:

$$F_3(x, y) + AF_2(x, y) + BF_1(x, y) = 0, \tag{1}$$

where A and B are the parameters of the system.

We think of the surface as given by

$$x_i = \phi_i(\mu, \nu) \qquad (i=1, 2, 3),$$

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† Annali di Matematica, vol. 7, 1866.

‡ Annali di Matematica, ser. 2, vol. 3, 1870.

§ For an independent proof of this proposition see F. BUSSE, *Inaugural Dissertation*, Berlin, 1896.

where μ and ν are rectangular surface coördinates, i. e., such that the systems of curves $\mu = \text{const.}$, $\nu = \text{const.}$, intersect orthogonally.

Let the relation between the surface and the plane be given by

$$x = \psi_1(\mu, \nu), \quad y = \psi_2(\mu, \nu).$$

Then we write $f_i(\mu, \nu)$ for $F_i(\psi_1, \psi_2)$ and equation (1) becomes

$$f_3(\mu, \nu) + Af_2(\mu, \nu) + Bf_1(\mu, \nu) = 0, \tag{2}$$

and from this we find:

$$f'_3(\mu, \nu) + Af'_2(\mu, \nu) + Bf'_1(\mu, \nu) = 0, \tag{3}$$

$$f'''_3(\mu, \nu) + Af'''_2(\mu, \nu) + Bf'''_1(\mu, \nu) = 0, \tag{4}$$

where

$$f'_i(\mu, \nu) = \frac{\partial f_i(\mu, \nu)}{\partial \mu} d\mu + \frac{\partial f_i(\mu, \nu)}{\partial \nu} d\nu,$$

$$f''_i(\mu, \nu) = \frac{\partial^2 f_i(\mu, \nu)}{\partial \mu^2} d\mu^2 + 2 \frac{\partial^2 f_i(\mu, \nu)}{\partial \mu \partial \nu} d\mu d\nu + \frac{\partial^2 f_i(\mu, \nu)}{\partial \nu^2} d\nu^2 + \frac{\partial f_i(\mu, \nu)}{\partial \mu} d^2\mu + \frac{\partial f_i(\mu, \nu)}{\partial \nu} d^2\nu.$$

From (2), (3) and (4) we obtain

$$\begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f'''_1 & f'''_2 & f'''_3 \end{vmatrix} = 0. \tag{5}$$

We write

$$n_{i1} = \frac{\partial f_i}{\partial \mu}, \quad n_{i2} = \frac{\partial f_i}{\partial \nu}; \quad m_{i1} = \frac{\partial^2 f_i}{\partial \mu^2}, \quad m_{i2} = \frac{\partial^2 f_i}{\partial \nu^2}, \quad m_{i3} = \frac{\partial^2 f_i}{\partial \mu \partial \nu}.$$

Making use of these abbreviations, substituting for f'_i and f'''_i their values, and expanding in terms of the first row, we have from equation (5):

$$f_1 \begin{vmatrix} n_{21}d\mu + n_{22}d\nu & m_{21}d\mu^2 + m_{22}d\nu^2 + 2m_{23}d\mu d\nu + n_{21}d^2\mu + n_{22}d^2\nu \\ n_{31}d\mu + n_{32}d\nu & m_{31}d\mu^2 + m_{32}d\nu^2 + 2m_{33}d\mu d\nu + n_{31}d^2\mu + n_{32}d^2\nu \end{vmatrix}$$

$$-f_2 \begin{vmatrix} n_{11}d\mu + n_{12}d\nu & m_{11}d\mu^2 + m_{12}d\nu^2 + 2m_{13}d\mu d\nu + n_{11}d^2\mu + n_{12}d^2\nu \\ n_{31}d\mu + n_{32}d\nu & m_{31}d\mu^2 + m_{32}d\nu^2 + 2m_{33}d\mu d\nu + n_{31}d^2\mu + n_{32}d^2\nu \end{vmatrix}$$

$$+f_3 \begin{vmatrix} n_{11}d\mu + n_{12}d\nu & m_{11}d\mu^2 + m_{12}d\nu^2 + 2m_{13}d\mu d\nu + n_{11}d^2\mu + n_{12}d^2\nu \\ n_{21}d\mu + n_{22}d\nu & m_{21}d\mu^2 + m_{22}d\nu^2 + 2m_{23}d\mu d\nu + n_{21}d^2\mu + n_{22}d^2\nu \end{vmatrix} = 0,$$

or

$$f_1\theta(23) - f_2\theta(13) + f_3\theta(12) = 0,$$

where

$$\theta(23) = \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} d\mu^3 + \begin{vmatrix} n_{21}m_{22} \\ n_{31}m_{32} \end{vmatrix} d\mu d\nu^2 + 2 \begin{vmatrix} n_{21}m_{23} \\ n_{31}m_{33} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{21}n_{22} \\ n_{31}n_{32} \end{vmatrix} d\mu d^2\nu$$
$$+ \begin{vmatrix} n_{22}m_{21} \\ n_{32}m_{31} \end{vmatrix} d\mu^2 d\nu + \begin{vmatrix} n_{22}m_{22} \\ n_{32}m_{32} \end{vmatrix} d\nu^3 + 2 \begin{vmatrix} n_{22}m_{23} \\ n_{32}m_{33} \end{vmatrix} d\mu d\nu^2 + \begin{vmatrix} n_{22}n_{21} \\ n_{32}n_{31} \end{vmatrix} d\nu d^2\mu ,$$

with corresponding expressions for $\theta(13)$, $\theta(12)$. Adding like terms, we have for terms in $d\mu^3$:

$$\left\{ f_1(\mu, \nu) \begin{vmatrix} n_{21}m_{21} \\ n_{31}m_{31} \end{vmatrix} - f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{31}m_{31} \end{vmatrix} + f_2(\mu, \nu) \begin{vmatrix} n_{11}m_{11} \\ n_{21}m_{21} \end{vmatrix} \right\} d\mu^3 ,$$

or

$$\begin{vmatrix} f_1(\mu, \nu) & n_{11} & m_{11} \\ f_2(\mu, \nu) & n_{21} & m_{21} \\ f_3(\mu, \nu) & n_{31} & m_{31} \end{vmatrix} d\mu^3 . \tag{6}$$

And we easily see that we have like determinants for the coefficients of the other terms. Indeed if we write [11] for the coefficient of $d\mu^3$, our differential equation is:

$$[11]d\mu^3 + [22]d\nu^3 + \{2[13] + [21]\}d\mu^2 d\nu + \{2[23] + [12]\}d\mu d\nu^2 + \Delta(d\mu d^2\nu - d\nu d^2\mu) = 0 , \tag{7}$$

where Δ is the determinant $|f_1 n_{21} n_{32}|$.

For shortness we write equation (7) in the form:

$$a_1 d\mu^3 + a_4 d\nu^3 + a_2 d\mu^2 d\nu + a_3 d\mu d\nu^2 + a_5(d\mu d^2\nu - d\nu d^2\mu) = 0 . \tag{8}$$

§ 2.

The geodetic lines of our surface are given by the differential equation:

$$\tfrac{1}{2} G \frac{\partial G}{\partial \mu} d\nu^3 - \tfrac{1}{2} E \frac{\partial E}{\partial \nu} d\mu^3 + \left[E \frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} \right] d\mu^2 d\nu$$
$$+ \left[G \frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} \right] d\mu d\nu^2 + EG(d\mu d^2\nu - d\nu d^2\mu) = 0 ; \tag{9}$$

we are to compare this with equation (8). This comparison gives the following system of partial differential equations:

$$E \frac{\partial G}{\partial \mu} - \tfrac{1}{2} G \frac{\partial E}{\partial \mu} = \lambda a_2 , \quad G \frac{\partial E}{\partial \nu} - \tfrac{1}{2} E \frac{\partial G}{\partial \nu} = -\lambda a_3 ,$$
$$\tfrac{1}{2} G \frac{\partial G}{\partial \mu} = \lambda a_4 , \quad \tfrac{1}{2} E \frac{\partial E}{\partial \nu} = -\lambda a_1 , \quad EG = \lambda a_5 . \tag{10}$$

From this system we eliminate λ by means of the last equation and obtain:

$$E\frac{\partial G}{\partial \mu} - \tfrac{1}{2}G\frac{\partial E}{\partial \mu} = EG\frac{a_2}{a_5}, \quad G\frac{\partial E}{\partial \nu} - \tfrac{1}{2}E\frac{\partial G}{\partial \nu} = -EG\frac{a_3}{a_5},$$

(11)

$$\tfrac{1}{2}G\frac{\partial G}{\partial \mu} = EG\frac{a_4}{a_5}, \quad \tfrac{1}{2}E\frac{\partial E}{\partial \nu} = -EG\frac{a_1}{a_5}.$$

From these we have:

1. $$\frac{\partial \log G}{\partial \mu} - \tfrac{1}{2}\frac{\partial \log E}{\partial \mu} = \frac{a_2}{a_5},$$

2. $$\frac{\partial \log E}{\partial \nu} - \tfrac{1}{2}\frac{\partial \log G}{\partial \nu} = -\frac{a_3}{a_5},$$

(12)

3. $$\frac{\partial \log E}{\partial \nu} = -2\frac{a_1}{a_5}\frac{G}{E},$$

4. $$\frac{\partial \log G}{\partial \mu} = -\frac{a_4}{a_5}\frac{E}{G}.$$

Integrating 1 and 2 we have

1. $$\log G - \log E = \log F_1 + \log V_1,$$

(13)

2. $$\log E - \tfrac{1}{2}\log G = -\log F_2 + \log U_1,$$

where $F_1 = e^{\int \frac{a_2}{a_5} d\mu}$ and $F_2 = e^{\int \frac{a_3}{a_5} d\mu}$; U_1 is a function of μ only and V_1 of ν only. Then (13) gives

$$\frac{G}{E^{\frac{1}{2}}} = F_1 V_1,$$

(14)

2. $$\frac{E}{G^{\frac{1}{2}}} = F_2^{-1} U_1.$$

Squaring each of these equations and multiplying it by the other we find

1. $$E = U_1^{\frac{2}{3}} V_1^{\frac{2}{3}} F_1^{\frac{1}{3}} F_2^{-\frac{1}{3}},$$

(15)

2. $$G = U_1^{\frac{1}{3}} V_1^{\frac{1}{3}} F_1^{\frac{2}{3}} F_2^{-\frac{2}{3}}.$$

From these last two equations we have the following values:

1. $$\frac{E}{G} = U_1^{\frac{1}{3}} V_1^{-\frac{1}{3}} F_1^{-\frac{1}{3}} F^{-}$$

(16) 2. $$\frac{\partial \log G}{\partial \mu} = \tfrac{2}{3}\frac{1}{U_1}\frac{dU_1}{d\mu} + \tfrac{4}{3}\frac{a_2}{a_5} - \tfrac{2}{3}\frac{\partial \log F_2}{\partial \mu},$$

3. $$\frac{\partial \log E}{\partial \nu} = \tfrac{2}{3}\frac{1}{V_1}\frac{dV_1}{\partial \nu} - \tfrac{4}{3}\frac{a_3}{a_3} + \tfrac{2}{3}\frac{\partial \log F_1}{\partial \nu}.$$

We substitute these values in equations (3) and (4) of the system (12) and obtain:

$$\frac{dU_1}{d\mu} + \left[2\frac{a_2}{a_5} - \frac{\partial \log F_2}{\partial \mu} \right] U_1 = 3\frac{a_4}{a_5} V_1^{-\frac{1}{3}} F_1^{-\frac{1}{3}} F_2^{-\frac{1}{3}} U_1^{\frac{1}{3}}, \tag{17}$$

$$\frac{dV_1}{d\nu} + \left[-2 \cdot \frac{a_3}{a_5} + \frac{\partial \log F_1}{\partial \nu} \right] V_1 = -3\frac{a_1}{a_5} U_1^{-\frac{1}{3}} F_1^{\frac{1}{3}} F_2^{\frac{1}{3}} V_1^{\frac{1}{3}}. \tag{18}$$

The integrals of these are:

$$1. \quad \frac{V_1^{\frac{2}{3}}}{U_1^{\frac{2}{3}}} = 2F_1^{\frac{1}{3}} F_2^{-\frac{1}{3}} (F_3 + V_2),$$

$$2. \quad \frac{U_1^{\frac{2}{3}}}{V_1^{\frac{2}{3}}} = 2F_1^{\frac{1}{3}} F_2^{-\frac{1}{3}} (F_4 + U_2), \tag{19}$$

where

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^2 d\nu$$

and U_2 and V_2 are functions of μ only and of ν only respectively.

These equations are *sufficient to say whether or not a solution exists and to find such solutions if they do exist.* It is proposed to illustrate this by an example (pp. 156–159) before going farther with the general theory.

EXAMPLE. We take $f_3 = f_3(\nu)$, a function of ν only; also

$$f_2(\mu, \nu) = \mu^n \nu^m, \quad f(\mu, \nu) = \mu^n \nu^\beta.$$

Then after some calculation we find the following values:

$$\begin{aligned}
a_1 &= 0, \\
a_2 &= \mu^{2n-2} \nu^{\beta+m-1} n(n+1)(\beta - m) f_3(\nu), \\
a_3 &= \mu^{2n-1} \nu^{\beta+m-2} \left[n(\beta - m)(m + \beta - 1) f_3(\nu) - 2n(\beta - m)\nu \frac{df_3(\nu)}{d\nu} \right], \\
a_4 &= \mu^{2n} \nu^{\beta+m-3} (\beta - m) \left[\beta m f_3(\nu) - (\beta + m - 1)\nu \frac{df_3(\nu)}{d\nu} + \nu^2 \frac{d^2 f_3(\nu)}{d\nu^2} \right], \\
a_5 &= \mu^{2n-1} \nu^{\beta+m-1} n(\beta - m) f_3(\nu).
\end{aligned} \tag{20}$$

From these we find:

$$\begin{aligned}
\frac{a_2}{a_5} &= \frac{n+1}{\mu}, \\
\frac{a_3}{a_5} &= \frac{m + \beta - 1}{\nu} - 2\frac{1}{f_3(\nu)} \frac{df_3(\nu)}{d\nu}, \\
\frac{a_1}{a_5} &= 0, \\
\frac{a_4}{a_5} &= \frac{\mu}{n} \left[\frac{\beta m}{\nu^2} - (\beta + m - 1)\frac{1}{\nu} \frac{1}{f_3(\nu)} \frac{df_3(\nu)}{d\nu} + \frac{1}{f_3(\nu)} \frac{d^2 f_3(\nu)}{d\nu^2} \right] = \frac{\mu}{n} \theta(\nu).
\end{aligned} \tag{21}$$

Then we have:

$$\int \frac{a_2}{a_5}\, d\mu = \int (n+1)\frac{d\mu}{\mu} = \log \mu^{n+1},$$

$$\int \frac{a_3}{a_5}\, d\nu = \int (m+\beta-1)\frac{d\nu}{\nu} - 2\int d\log f_3(\nu) d\nu = \log \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}.$$

Therefore

$$F_1 = \mu^{n+1}, \quad F_2 = \frac{\nu^{m+\beta-1}}{f_3^2(\nu)}, \quad F_4 = 0,$$

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\theta(\nu)\mu^{-2(n+1)+1} d\mu = \frac{\theta(\nu)}{2n^2}\mu^{-2n}.$$

Introducing these values into (19) we have:

(22)

1. $$V_1^{\frac{3}{2}} U_1^{-\frac{3}{2}} = 2\mu^{\frac{3}{2}(n+1)}\nu^{-\frac{3}{2}(m+\beta-1)} f_3^{\frac{3}{2}}(\nu)\left[\frac{\theta(\nu)\mu^{-2n} + 2n^2 V_2}{2n^2}\right]$$

2. $$V_1^{-\frac{3}{2}} U_1^{\frac{3}{2}} = 2\mu^{\frac{3}{2}(n+1)}\nu^{-\frac{3}{2}(m+\beta-1)} f_3^{\frac{3}{2}}(\nu) U_2.$$

This requires either

(a) $$V_2 = K\theta(\nu) \qquad (K = \text{constant}),$$

or

(b) $$\theta(\nu) = 0.$$

Case (a). If $V_2 = K\theta(\nu)$, then we find:

$$U_1^{\frac{3}{2}} = \lambda\mu^{-\frac{3}{2}(n+1)}[\mu^{-2n} + 2n^2 K]^{-1},$$

$$U_1^{\frac{3}{2}} = a\mu^{\frac{3}{2}(n+1)} U_2,$$

which require that

$$U_2 = \frac{\lambda}{a}\mu^{-2(n+1)}[\mu^{-2n} + 2n^2 K]^{-1};$$

therefore

$$U_1 = \frac{\lambda^{\frac{2}{3}}\mu^{n-2}}{[2n^2 K\mu^{2n} + 1]^{\frac{2}{3}}}.$$

And for V_1 we find:

$$V_1^{\frac{3}{2}} = \lambda'\nu^{-\frac{3}{2}(m+\beta-1)} f_3^{\frac{3}{2}}(\nu)\theta(\nu),$$

$$V_1^{\frac{3}{2}} = a'\nu^{\frac{3}{2}(m+\beta-1)} f_3^{-\frac{3}{2}}(\nu).$$

These require that:

$$\theta(\nu) = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)};$$

Taking $a' = \lambda'$, as we may do, and restoring the value of $\theta(\nu)$ we have the following differential equation to determine $f_3(\nu)$:

$$(23)\qquad \frac{d^2f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \beta m\nu^{-2}f_3(\nu) = \nu^{2m+\beta-1}f_3^{-3}(\nu).$$

Before solving this differential equation we consider case (b). If $\theta(\nu) = 0$ we find:

$$U_1^{\frac{3}{2}} = \lambda\mu^{-\frac{3}{2}(n+1)},$$

$$U_1^{\frac{3}{2}} = a\mu^{\frac{3}{2}(n+1)}U_2,$$

giving

$$U_2 = \frac{\lambda}{\mu}\mu^{-2(n+1)};$$

also

$$V_1^{\frac{3}{2}} = \lambda'\nu^{-\frac{3}{2}(m+\beta-1)}f_3^{\frac{3}{2}}(\nu)\,V_2,$$

$$V_1^{\frac{3}{2}} = a'\nu^{\frac{3}{2}(m+\beta-1)}f_1^{-\frac{3}{2}}(\nu),$$

giving

$$V_2 = \frac{a'}{\lambda'}\frac{\nu^{2(m+\beta-1)}}{f_3^4(\nu)}.$$

And $f_3(\nu)$ is given by

$$(24)\qquad \frac{d^2f_3(\nu)}{d\nu^2} - (\beta + m - 1)\nu^{-1}\frac{df_3(\nu)}{d\nu} + \frac{\beta m}{\nu^2}f_3(\nu) = 0.$$

It remains then to see if $f_3(\nu)$ can be determined algebraically.

Since equation (24) is (23) with its right hand number put equal to zero we can consider them together.

We put

$$\nu = e^{\vartheta},\quad f_3(\nu) = e^{\frac{(m+\beta)\vartheta}{2}}z,\quad \theta = \frac{dz}{d\vartheta}.$$

After some reduction equations (23) and (24) reduce to the forms:

$$(25)\qquad \theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4}z = z^{-3},$$

$$(26)\qquad \theta\frac{d\theta}{dz} - \frac{(m-\beta)^2}{4}z = 0.$$

We consider the latter and simpler case first.

It gives:

$$\theta = \tfrac{1}{2}\sqrt{(m-\beta)^2z^2 + 4c}.$$

Hence

$$\vartheta = \int\frac{2dz}{\sqrt{(m-\beta)^2z^2 + 4c}}.$$

Therefore

$$\vartheta = \log\left[\frac{2(m-\beta)^2z + 2(m-\beta)\sqrt{(m-\beta)^2z^2 + 4c_1}}{2(m-\beta)^2z - 2(m-\beta)\sqrt{(m-\beta)^2z^2 + 4c_1}}\right]^{\frac{1}{m-\beta}} + \log c_2.$$

Remembering that $\vartheta = \log \nu$ and that $z = f_3(\nu)/\nu^{\frac{m+\beta}{2}}$, we find after some reduction:

$$f_3^2(\nu) = \frac{4c_1\nu^{m+\beta}}{(m-\beta)^2\{(\nu^{m-\beta}-c_2)^2-1\}};$$

and for the equation of the curve:

$$(m-\beta)^2\{(\nu^{m-\beta}-c_3)^2-1\}(A\mu^n\nu^m + B\mu^n\nu^\beta)^2 - 4c_1\nu^{m+\beta} = 0.$$

Hence

$$V_1 = K_1[\nu^{2(m-\beta)-1} - 2c_2\nu^{m+\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

$$U_1 = K_2\mu^{-2(n+1)} \qquad (K_1 \text{ and } K_2 \text{ constant});$$

these give:

$$E = p\mu^{-2(n+1)},$$

$$G = q[\nu^{2(m-\beta)-1} - 2c_2\nu^{m-\beta-1} + (c_2^2-1)\nu^{-1}]^2,$$

where p and q are constant.

In regard to the other integral:

$$\vartheta = \int \frac{2zdz}{2cz + \sqrt{(m-\beta)^2z^4 - 4}},$$

it appears that in order to keep the exponents rational and the coefficients real it is necessary to take $c = 0$.

Then we find finally:

$$f_3^2(\nu) = \frac{4K^2\nu^{2\beta} + \nu^{2m}}{2K(m-\beta)},$$

where K is constant, and the curve is given by the equation:

$$4K^2\nu^{2\beta} + \nu^{2m} = 2K(m-\beta)(A\mu^n\nu^m + B\mu^n\nu^\beta)^2.$$

To revert to the general theory, we may say that if a solution exists, the right hand member of equation 1 of (19) must be of the form

$$\frac{\Theta(\nu)}{\Theta(\mu)},$$

and then

$$V_1^3 = K\Theta(\nu), \quad U_1^3 = K\Theta(\mu).$$

The functions $\psi_1(\mu, \nu)$ and $\psi_2(\mu, \nu)$ are given by the equations:

$$\frac{F_1(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_1(\mu, \nu)}{f_3(\mu, \nu)}, \qquad \frac{F_2(\psi_1, \psi_2)}{F_3(\psi_1, \psi_2)} = \frac{f_2(\mu, \nu)}{f_3(\mu, \nu)}.$$

Since we desire a conformal representation we shall take

$$\psi_1 = \mu, \quad \psi_2 = \nu.$$

But it is plain that by such a choice certain conditions are imposed by equations (19) and we proceed to consider those conditions.

§ 3.

Calculation of V_2 and U_2.

If we represent the right hand members (19) by R and S respectively, then we must have:

1. $$RS = 1,$$

(27) 2. $$\frac{\partial^2 \log R}{\partial \mu \partial \nu} = 0,$$

3. $$\frac{\partial^2 \log S}{\partial \mu \partial \nu} = 0.$$

From the second of these we find:

$$(28) \qquad \tfrac{4}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} - \tfrac{2}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} + \frac{\partial^2 \log (F_3 + V_2)}{\partial \mu \partial \nu} = 0.$$

Considering the last term of this equation we find

$$\frac{\partial^2 \log (F_3 + V_2)}{\partial \mu \partial \nu} = \frac{\partial}{\partial \nu}\frac{\partial F_3/\partial \nu}{F_3 + V_2} = \frac{\partial}{\partial \nu}\left\{-4\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2)\right\}$$

$$= -4F_2^{-2}(F_4 + U_2)\frac{\partial}{\partial \nu}\frac{a_4}{a_5} + 8\frac{a_3}{a_5}\frac{a_4}{a_5}F_2^{-2}(F_4 + U_2) - 4\frac{a_1}{a_5}\frac{a_4}{a_5},$$

since

$$\frac{\partial F_3}{\partial \mu} = -\frac{a_4}{a_5}F_1^{-2} \quad \text{and} \quad 4F_1^2F_2^{-2}(F_3 + V_2)(F_4 + U_2) = 1.$$

Whence from (28) we find:

$$(29) \qquad 2F_2^{-2}(F_4 + U_2) = \frac{\frac{2}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} - \frac{4}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_2}{a_5}\frac{a_4}{a_5} - 2\frac{\partial}{\partial \nu}\frac{a_4}{a_5}};$$

and from 3 of (27) we find in like manner:

$$(30) \qquad 2F_1^2(F_3 + V_2) = \frac{\frac{4}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} - \frac{2}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} + 4\frac{a_1}{a_5}\frac{a_4}{a_5}}{4\frac{a_1}{a_5}\frac{a_2}{a_5} + 2\frac{\partial}{\partial \mu}\frac{a_1}{a_5}}.$$

If we represent the right hand members of (29) and (30) by Δ_1 and Δ_2 respectively, then equations (19) become:

$$\frac{V_1^{\frac{1}{3}}}{U_1^{\frac{1}{3}}} = F_1^{-\frac{1}{3}} F_2^{-\frac{1}{3}} \Delta_1,$$

(31)

$$\frac{U_1^{\frac{1}{3}}}{V_1^{\frac{1}{3}}} = F_1^{\frac{1}{3}} F_2^{\frac{1}{3}} \Delta_2.$$

§ 4.

Partial differential equations satisfied by $f_i(\mu, \nu)$.

As a first condition we have

$$\Delta_1 \Delta_2 = 1.$$

Differentiating (29) with respect to ν, and using (19) we have

$$2 \frac{a_1}{a_5} F_2^2 = F_2^2 \frac{\partial \Delta_1}{\partial \nu} + 2 F_2^2 \frac{a_3}{a_5} \Delta_1,$$

or

$$\frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 = 2 \frac{a_1}{a_5};$$

and in like manner from (30) we find

$$\frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 = -2 \frac{a_4}{a_5}.$$

Hence we have the system:

$$\Delta_1 \Delta_2 = 1,$$

(32)

$$\frac{\partial \Delta_1}{\partial \nu} + 2 \frac{a_3}{a_5} \Delta_1 = 2 \frac{a_1}{a_5},$$

$$\frac{\partial \Delta_2}{\partial \mu} - 2 \frac{a_2}{a_5} \Delta_2 = -2 \frac{a_4}{a_5}.$$

These are the necessary, and, as is easily seen, sufficient conditions to impose upon $f_i(\mu, \nu)$ in order that a solution may exist for the rectangular coördinate system (μ, ν).

§ 5.

We consider next a somewhat general solution of our problem. We have written

$$\int \frac{a_2}{a_5} d\mu = \log F_1, \quad \int \frac{a_3}{a_5} d\nu = \log F_2.$$

We consider one case where these integrals actually yield logarithms, i. e., $\partial a_5/\partial \mu = \lambda a_2$ and $\partial a_5/\partial \nu = a a_3$. To simplify the expressions for a_i we divide equation

(2) through by $f_2(\mu, \nu)$ and then consider $f_3' = 1$, $f_2' = f_2/f_3$, $f_3' = f_1/f_3$. This amounts to putting $f_3 = 1$ in our formulas. We find:

$$a_1 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_4 = \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}, \quad a_5 = \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial f_3}{\partial \nu} \end{vmatrix},$$

$$a_2 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu^2} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu^2} \end{vmatrix}, \quad a_3 = 2\begin{vmatrix} \frac{\partial f_2}{\partial \nu} & \frac{\partial^2 f_2}{\partial \mu \partial \nu} \\ \frac{\partial f_3}{\partial \nu} & \frac{\partial^2 f_3}{\partial \mu \partial \nu} \end{vmatrix} + \begin{vmatrix} \frac{\partial f_2}{\partial \mu} & \frac{\partial^2 f_2}{\partial \nu^2} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial^2 f_3}{\partial \nu^2} \end{vmatrix}.$$

The case to be considered is found by taking $\partial^2 f_2/\partial\mu\partial\nu = 0$ and $\partial^2 f_3/\partial\mu\partial\nu = 0$, that is, $f_2 = \phi_1(\mu) + \phi_2(\nu)$ and $f_3 = \psi_1(\mu) + \psi_2(\nu)$. Then we shall have:

$$a_5 = \phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu),$$
$$a_1 = \phi_1'(\mu)\,\psi_1''(\mu) - \psi_1'(\mu)\,\phi_1''(\mu),$$
$$a_4 = \phi_2'(\nu)\,\psi_2''(\nu) - \psi_2'(\nu)\,\phi_2''(\nu),$$
$$a_2 = \phi_2'(\nu)\,\psi_1''(\mu) - \psi_2'(\nu)\,\phi_1''(\mu),$$
$$a_3 = \phi_1'(\mu)\,\psi_2''(\nu) - \psi_1'(\mu)\,\phi_2''(\nu).$$

Hence

$$\int \frac{a_2}{a_5}\,d\mu = -\log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\}, \tag{40}$$

$$\int \frac{a_3}{a_5}\,d\nu = \log\{\phi_1'(\mu)\,\psi_2'(\nu) - \psi_1'(\mu)\,\phi_2'(\nu)\}, \tag{41}$$

Therefore

$$F_1 = 1 \div \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = \frac{1}{a_5}, \qquad F_2 = \begin{vmatrix} \phi_1'(\mu) & \phi_2'(\nu) \\ \psi_1'(\mu) & \psi_2'(\nu) \end{vmatrix} = a_5.$$

Hence

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2}\,d\mu = -a_4 \int a_5\,d\mu = -\begin{vmatrix} \phi_1(\mu) & \phi_2'(\nu) \\ \psi_1(\mu) & \psi_2'(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix},$$

$$F_4 = \int \frac{a_1}{a_5} F_2^2\,d\nu = a_1 \int a_5\,d\nu = \begin{vmatrix} \phi_1'(\mu) & \phi_2(\nu) \\ \psi_1'(\mu) & \psi_2(\nu) \end{vmatrix} \cdot \begin{vmatrix} \phi_1'(\mu) & \phi_1''(\mu) \\ \psi_1'(\mu) & \psi_1''(\mu) \end{vmatrix}.$$

In order to separate the variables it is necessary that either

$$\text{(a)} \quad \phi_1(\mu) = \psi_1(\mu) \quad \text{or} \quad \text{(b)} \quad \phi_2(\nu) = \psi_2(\nu),$$

and since the equations and expressions are symmetrical in μ and ν it will only be necessary to consider case (a) and afterwards interchange μ and ν in the result.

First of all we have:

$$F_4 = 0, \quad F_1 = \frac{1}{\phi'(\mu)} \frac{1}{\psi_2'(\nu) - \phi_2'(\nu)}, \quad F_2 = \phi'(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\},$$

$$F_3 = -\phi(\mu)\,\{\psi_2'(\nu) - \phi_2'(\nu)\} \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}.$$

Then from (19) we have:

$$(42) \qquad \frac{\frac{1}{2}\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}{\dfrac{V_2}{\{\psi_2'(\nu) - \phi_2'(\nu)\}} - \phi(\mu) \begin{vmatrix} \phi_2'(\nu) & \phi_2''(\nu) \\ \psi_2'(\nu) & \psi_2''(\nu) \end{vmatrix}} = \frac{2U_2}{\phi'^2(\mu)\{\psi_2'(\nu) - \phi_2'(\nu)\}}.$$

As V_1 must vanish and $U_2 = \phi_1'^4(\mu)/4\phi(\mu)$, we find

$$U_1 = \frac{1}{\sqrt{8}} \frac{\phi'^3(\mu)}{\phi^{\frac{3}{2}}(\mu)}.$$

Also

$$\begin{vmatrix} \phi_2'''(\nu) & \phi_2'(\nu) \\ \psi_2'''(\nu) & \psi_2'(\nu) \end{vmatrix} = \{\psi_2'(\nu) - \phi_2'(\nu)\}^3,$$

or, if we write x for $\psi_2'(\nu)$ and y for $\phi_2'(\nu)$,

$$x\frac{dy}{d\nu} - y\frac{dx}{d\nu} = (x - y)^3.$$

Put $x - y = q$ and our differential equation is:

$$q\frac{dx}{d\nu} - x\frac{dq}{d\nu} = q^3,$$

or

$$\frac{d}{d\nu}\frac{x}{q} = q.$$

Therefore

$$\frac{x}{q} = \int q\,d\nu = \psi_2(\nu) - \phi_2(\nu) + c;$$

hence

$$\frac{\psi_2'(\nu)}{\psi_2'(\nu) - \phi_2'(\nu)} = \psi_2(\nu) - \phi_2(\nu) + c_1,$$

and integrating this we have:

$$\psi_2(\nu) = \tfrac{1}{2}\,[\psi_2(\nu) - \phi_2(\nu)]^2 + c_1[\psi_2(\nu) + \phi_2(\nu)] + c_2.$$

Remembering that $\phi_2(\nu) = f_1(\nu)/f_3(\nu)$ and $\phi(\mu) = f_1(\mu)/f_3(\mu)$ we have, after some reduction:

$$B^2[K_2 f_1(\nu) + K_3 f_3(\nu)]\, f_3(\nu) f_3^2(\mu) = \{(K_1 B + 1) f_3(\nu) f_3(\mu) + (A + B)[f_1(\mu) f_3(\nu) + f_3(\mu) f_1(\nu)]\}^2, \tag{43}$$

for the equation of our curve. Here the f_i of μ and of ν are *any algebraic functions whatever*. As mentioned above we find a solution for case (b) by interchanging μ and ν in (43).

§ 6.

Conditions for the equality of E and G.

The work thus far is for $F = 0$, i. e., μ and ν are rectangular surface coördinates. It remains to consider the case $E = G$.

From our system of partial differential equations (10) we find that a_1 must be equal to a_3 and a_2 to a_4. It remains to consider the system of equations (32) which were the necessary and sufficient conditions to impose on the $f_i(\mu, \nu)$. First we consider the values of F_3 and F_4. We recall that

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu, \quad F_4 = \int \frac{a_1}{a_5} F_2^{-2} d\nu,$$

$$\log F_1 = \int \frac{a_2}{a_5} d\nu, \quad \log F_2 = \int \frac{a_3}{a_5} d\nu.$$

Hence also

$$\frac{\partial F_1}{\partial \mu} = \frac{a_2}{a_5} F_1, \quad \frac{\partial F_2}{\partial \nu} = \frac{a_3}{a_5} F_2.$$

We have then

$$F_3 = \int -\frac{a_4}{a_5} F_1^{-2} d\mu = \int -\frac{a_2}{a_5} F_1^{-2} d\mu = \int -F_1^{-3} \frac{\partial F_1}{\partial \mu} d\nu = \tfrac{1}{2} F_1^{-2}.$$

Also

$$F_4 = \int \frac{a_3}{a_5} F_2^2 d\nu = \int F_2 \frac{\partial F_2}{\partial \nu} d\nu = \tfrac{1}{2} F_2^2.$$

Hence the right hand members of (29) and (30) become respectively,

$$2F_2^{-2}(\tfrac{1}{2}F_2^2 + U_2) = 1 + 2F_2^{-2} U_2 = \Delta_1 = \frac{\frac{2}{3}\frac{\partial}{\partial \mu}\frac{a_3}{a_5} - \frac{4}{3}\frac{\partial}{\partial \nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} - 2\frac{\partial}{\partial \nu}\frac{a_2}{a_5}},$$

$$2F_1^2(\tfrac{1}{2}F_1^{-2} + V_2) = 1 + 2F_1^2 V_2 = \Delta_2 = \frac{\frac{4}{3}\frac{\partial}{\partial\mu}\frac{a_3}{a_5} - \frac{2}{3}\frac{\partial}{\partial\nu}\frac{a_2}{a_5} + 4\frac{a_2}{a_5}\frac{a_3}{a_5}}{4\frac{a_2}{a_5}\frac{a_3}{a_5} + 2\frac{\partial}{\partial\mu}\frac{a_3}{a_5}}.$$

We consider the first of equations (3), i. e., $\Delta_1\Delta_2 = 1$.

We write

$$\Delta_1 = \frac{\frac{2}{3}a - \frac{4}{3}b + c}{c - 2b} = 1 + \frac{\frac{2}{3}(a+b)}{c-2b},$$

and also

$$\Delta_2 = \frac{\frac{4}{3}a - \frac{2}{3}b + c}{c + 2a} = 1 - \frac{\frac{2}{3}(a+b)}{c+2a}.$$

Therefore

$$\Delta_1\Delta_2 = 1 + \frac{\frac{2}{3}(a+b)}{c-2b} - \frac{\frac{2}{3}(a+b)}{c+2a} - \frac{\frac{4}{9}(a+b)^2}{(c+2a)(c-2b)}$$

$$= 1 + \frac{\frac{2}{3}(a+b)}{(c+2a)(c-2b)}\left[c + 2a - c + 2b - \tfrac{2}{3}(a+b)\right]$$

$$= 1 + \frac{\frac{8}{9}(a+b)}{(c+2a)(c-2b)}.$$

Hence we must have $a + b = 0$ and therefore

$$\frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} = 0.$$

Hence $\Delta_1 = 1$ and $\Delta_2 = 1$. Then the other two equations of (32) are satisfied and we may conclude that

$$(44)\qquad \begin{aligned} a_1 &= a_3, \\ a_2 &= a_4, \\ \frac{\partial}{\partial\mu}\frac{a_3}{a_5} + \frac{\partial}{\partial\nu}\frac{a_2}{a_5} &= 0, \end{aligned}$$

are the necessary and sufficient conditions that $E = G$.

The next question will be the study of these surfaces, their existence and properties. In particular I have already finished part of the work for the curve $y^2 = ax^3 + bx^2 + cx + d$, i. e., for surfaces whose geodetic lines go into cubics upon the plane by a conformal transformation. The straight line and circle lead to the surfaces of constant curvature with their interesting properties. Here is a much wider and more interesting field.

GÖTTINGEN, *November* 3, 1900.

ON THE EXISTENCE OF A MINIMUM OF THE INTEGRAL $\int_{x_0}^{x_1} F(x, y, y')\,dx$ WHEN x_0 AND x_1 ARE CONJUGATE POINTS, AND THE GEODESICS ON AN ELLIPSOID OF REVOLUTION: A REVISION OF A THEOREM OF KNESER'S*

BY

W. F. OSGOOD

In a paper entitled *Zur Variationsrechnung* KNESER enunciates the theorem † that the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

ceases to be a minimum, not only when the interval (x_0, x_1) contains a point x' conjugate ‡ to x_0, but when x_1 coincides with x'. This theorem is true in general, but not in all cases, and it is the object of this paper to correct and extend the theorem and to give a complete solution of the problem that KNESER proposed to himself. The geodesics on an ellipsoid of revolution afford an example of a case in which the theorem as enunciated by KNESER is not correct.

The methods and results of this paper admit of extension to multiple integrals.

The contents are as follows: §§ 1, 2 are introductory; § 3 gives an account of KNESER'S memoir and obtains his results in a simple manner; §§ 4–6 contain the new material of the paper.

§ 1. *Necessary Conditions.*

A necessary condition that the integral I be made a minimum by the function y is that y satisfy LAGRANGE'S equation

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† Mathematische Annalen, vol. 50 (1897), p. 50. Cf. also § 6 of this paper. This theorem had however been stated and proved for the general case twenty years earlier by ERDMANN: Zeitschrift für Mathematik und Physik, vol. 22 (1877), p. 324, who showed that, when $\delta y = \varepsilon \phi_\gamma(x, \gamma_0)$ (in the notation of the present paper) the second variation of the integral I vanishes while

$$\delta^2 I = -\varepsilon^3 (F_{y'y'})_{x=x'} \phi'_\gamma(x', \gamma_0)\phi_{\gamma\gamma}(x', \gamma_0),$$

and this is not in general zero. For this reference I am indebted to Professor BOLZA.

‡ The meaning of the word *conjugate* will appear below, § 3.

$$F_y - \frac{dF_{y'}}{dx} = 0, \qquad \left(F_y = \frac{\partial F}{\partial y}, \text{ etc.}\right) \tag{1}$$

or

$$F_y - F_{xy'} - y' F_{yy'} - y'' F_{y'y'} = 0,$$

and furthermore fulfill the boundary conditions

$$y(x_0) = y_0, \quad y(x_1) = y_1,$$

where y_0, y_1 are fixed quantities.* Here the function $F(x, y, p)$, regarded as a function of the three independent variables (x, y, p), is to be continuous, together with its partial derivatives of the first and second orders, throughout a region R:

$$A_1 \leqq x \leqq B_1, \quad A_2 < y < B_2, \quad A_3 < p < B_3,$$

and $A_1 \leqq x_0 < x_1 \leqq B_1$. For simplicity we will assume $A_3 = -\infty$, $B_3 = +\infty$, and require that $F_{y'y'}(x, y, p)$ shall vanish at no point of R. The function $F_{y'y'}(x, y, p)$ will, then, not change sign in R.

The functions y admitted to consideration are single valued and continuous throughout the interval $x_0 \leqq x \leqq x_1$ and assume the prescribed boundary values, and they have a first derivative which, at least within the interval $x_0 < x < x_1$, is finite and continuous.

We will denote with KNESER† any solution of (1), or the curve that represents such a function, as an *extremal*.

§ 2. *Sufficient Conditions.*‡

WEIERSTRASS'S sufficient condition that such a function y, represented by a curve C, actually makes I a minimum consists (a) in the existence of a field of extremals surrounding C, and (b) in the relation $F_{y'y'}(x, y, p) > 0$, where (x, y) is any point of C and p may have any value. The notion of the field was first published by SCHWARZ.§ That which is essential may be embraced in the following requirements:

* We confine ourselves for simplicity to the case of integrals with fixed limits; but the method here set forth applies equally well to integrals with variable limits. In fact, one of the first examples that I formed was the following. Reversing the usual order of the processes, I started with the envelope, which I assumed as the semicubical parabola $y^2 = x^3$, and took then as extremals the right lines that envelop it. I then constructed a problem of the calculus of variations, of which these elements afford the solution and which, since the above family of right lines is cut orthogonally by a certain parabola P $(y^2 = ax + \beta)$, may be formulated as follows: *To find the shortest line that can be drawn from the point $x = 0$, $y = 0$ to the parabola P.*

† *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, p. 24.

‡ For an exposition of WEIERSTRASS'S sufficient condition for a minimum, together with KNESER'S and HILBERT'S proofs of its validity, cf. an article by the writer, Annals of Mathematics, ser. 2, vol. 2, no 3 (1901), p. 105.

§ Festschrift on the occasion of WEIERSTRASS'S seventieth birthday: *Ueber ein die Flächen kleinsten Inhalts betreffendes Problem der Variationsrechnung*, Acta soc. sci. Fennicae, vol. 15 (1885), p. 315 = Werke, vol. 1, p. 223. Cf. also KNESER, *Variationsrechnung*, chap. 3.

(1) Through each point (x, y) of the neighborhood S of the extremal C there passes one and only one extremal

$$y = \phi(x, \gamma),$$

γ being a parameter. The function ϕ, regarded as a function of the two independent variables (x, γ), is together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi'_\gamma$ continuous at each point (x, γ) corresponding to an interior point (x, y) of S; and $\phi'(x, \gamma)$ remains finite throughout that part of the neighborhoods of (x_0, y_0), (x_1, y_1) that is bounded by any two curves of the class admitted to consideration in § 1. The equation of C is $y = \phi(x, \gamma_0)$.

(2) The function $\phi_\gamma(x, \gamma) \neq 0$, when $x_0 < x < x_1$.

By the "neighborhood" of the extremal C is meant the interior of a strip S of the xy-plane bounded by the lines $x = x_0$, $x = x_1$, $y = \phi(x, \gamma_0) + \epsilon$, $y = \phi(x, \gamma_0) - \epsilon$, where ϵ is a positive constant, together with the points (x_0, y_0), (x_1, y_1).

The definition of the field here given is broader than the one employed by KNESER in his treatise or the one that I used in the paper above referred to. Both KNESER's and HILBERT's proofs for WEIERSTRASS's sufficient condition, given in the latter article, can be extended without difficulty to the present case. Thus, in the case of HILBERT's proof, it is sufficient to show that the value of the integral

$$J = \int_{x_0}^{x_1} \{F(x, Y, y') + (Y' - y') F_{y'}(x, Y, y')\} dx,$$

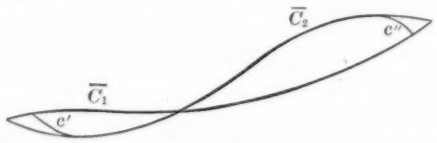
where x, Y are the coördinates of a point of an arbitrary curve C lying in the field, and where

$$y' = \phi'(x, \gamma), \quad Y = \phi(x, \gamma),$$

is independent of C, i. e., is the same for any two such curves, C_1, C_2, and this can be proved as follows. Let the values of J for C_1 and C_2 be denoted respectively by J_1 and J_2, and suppose

$$J_1 - J_2 = h \gtrless 0.$$

Let (x', y') and (x'', y'') be two points of C_1 chosen respectively arbitrarily near to the points (x_0, y_0) and (x_1, y_1); and let C' be a curve connecting (x', y')



and (x'', y'') and coinciding throughout the greater part of its extent with C_2. More precisely, let curves c' and c'' be drawn respectively from (x', y') and

(x'', y'') tangent to C_2 and meeting C_2 at points that lie near to (x', y') and (x'', y'') respectively, and let c' and c'' be so chosen that the curve C', consisting of c', c'' and the part of C_2 lying between the points of tangency of c' and c'', will be a curve of the class admitted to consideration, § 1. Denote the values of J taken along C_1 from (x', y') to (x'', y'') and along C' respectively by J_1' and J'. Then

$$J_1' - J' = 0.$$

For, all of the conditions of the lemma on which HILBERT's proof rests are fulfilled for the interval $x' \leqq x \leqq x''$. Now it is clear that, by a proper choice of the points (x', y'), (x'', y'') and the curves c', c'', the integrals J_1' and J' can be made to differ respectively from J_1 and J_2 by quantities each numerically less than $\frac{1}{2}h$. For, though the function $y' = \phi'(x, \gamma)$ may become infinite in the neighborhood of the point (x_0, y_0), still it is possible to cut out from S an angle bounded by the lines $y - y_0 = \pm\lambda(x - x_0)$, where λ is an arbitrarily chosen constant, within which $\phi'(x, \gamma)$ remains finite and from which the curves C_1, C_2, C' never emerge; and since a similar remark applies to the neighborhood of the point (x_1, y_1), this is sufficient for the proof. Hence the supposition that $h > 0$ leads to a contradiction, and

$$J_1 = J_2.$$

§ 3. *Kneser's Investigation.**

KNESER's results may be briefly obtained as follows. Let there exist an extremal C connecting the points (x_0, y_0) and (X_1, Y_1). Consider the integral I formed for an arbitrary value of x_1 between x_0 and X_1, the point (x_1, y_1) lying on C. Further assume

(1) That through the point (x_0, y_0) and each point (x, y) of that part of the neighborhood of (x_0, y_0) for which $x > x_0$ (i. e., $x_0 < x < x_0 + \mu$, $|y - y_0| < \mu$), one and only one extremal $y = \phi(x, \gamma)$ passes and that $y = \phi(x, \gamma_0)$ coincides with C. The function $\phi(x, \gamma)$, regarded as a function of the two independent variables (x, γ), shall, together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi_\gamma'$, be continuous at each of the points (x, γ) corresponding to points (x, y) of the above neighborhood; and furthermore at the points (x, γ) of the region

$$x_0 + \mu \leqq x \leqq X_1, \qquad \gamma_0 - \kappa < \gamma < \gamma_0 + \kappa:$$

and $\phi'(x, \gamma)$ shall remain finite throughout that part of the neighborhood of (x_0, y_0) bounded by any two curves of the class considered in § 1.

(2) $\phi_\gamma(x, \gamma) \neq 0$ when $x_0 < x < x_0 + \mu$, $|y - y_0| < \mu$;

$\phi_\gamma(x, \gamma_0) \neq 0$ when $x_0 + \mu \leqq x < x' < X_1$; $\phi_\gamma(x', \gamma_0) = 0$.

* Mathematische Annalen, loc. cit.

Since all the extremals $y = \phi(x, \gamma)$ go through the point (x_0, y_0) it follows that $\phi_\gamma(x_0, \gamma) \equiv 0$.

When $x_1 < x'$, a field exists about the extremal C, as is readily proved by the ordinary methods of analysis.* We will assume once for all that $F_{y'y'}(x, y, p)$ is positive, and hence the integral I is a minimum.

When $x_1 = x'$, x' is called the *conjugate point* of x_0, and it is the study of this case to which KNESER's paper is devoted. It is desirable to distinguish three cases. We will assume that $\phi_{\gamma\gamma}(x, \gamma)$ exists and is continuous throughout the neighborhood of the point (x', γ_0).

Case I. $\phi_{\gamma\gamma}(x', \gamma_0) \neq 0$. This is the general case.

Case II. $\phi_{\gamma\gamma}(x', \gamma) \equiv 0$, $|\gamma - \gamma_0| < \delta$.

Case III. $\phi_{\gamma\gamma}(x', \gamma_0) = 0$, $\phi_{\gamma\gamma}(x', \gamma) \not\equiv 0$.

Cases I and II are considered by KNESER and his results are as follows. In Case I the extremals $y = \phi(x, \gamma)$ have an envelope † which passes through the point x', $y' = \phi(x', \gamma_0)$ and this point is an ordinary point of the envelope. The proof may be based on DINI's existence theorem for implicit functions of real variables,‡ according to which the equation

$$0 = \phi_\gamma(x, \gamma)$$

can be solved for γ in the neighborhood of the point (x', γ_0). The function

$$\gamma = \psi(x)$$

thus defined is single valued in the neighborhood of the point x' and it has a continuous derivative throughout this region, given by the formula

$$\frac{d\gamma}{dx} = -\frac{\phi'_\gamma(x, \gamma)}{\phi_{\gamma\gamma}(x, \gamma)}.$$

Moreover, the pairs of values $(x, \gamma = \psi(x))$ are the only pairs of values (x, γ) in the neighborhood of the point (x', γ_0) which cause $\phi_\gamma(x, \gamma)$ to vanish.

The equation of the envelope is

$$y = \phi(x, \gamma) \qquad \text{where} \quad \gamma = \psi(x).$$

Its slope is given by the formula

$$\frac{dy}{dx} = \phi'(x, \gamma) + \phi_\gamma(x, \gamma)\frac{-\phi'_\gamma(x, \gamma)}{\phi_{\gamma\gamma}(x, \gamma)} = \phi'(x, \gamma).$$

* The details of the proof are given in a similar case in the writer's paper, Annals of Mathematics, loc. cit., § 2.

† That this *in general* is the case is essentially JACOBI's criterion. The introduction of sufficient restrictions so that one is able to prove that, in the problem thus restricted, an envelope does exist which has an ordinary point at (x', γ_0), is characteristic of the progress of analysis since Jacobi's time.

‡ Cf. PEANO-GENOCCHI, *Calcolo differenziale*, etc., § 110, or JORDAN, *Cours d'Analyse*, vol. 1, 2d ed., p. 80. KNESER uses the method of power series.

Thus in Case I an envelope in the proper sense of the term exists, and it *enters the interval* (x_0, x'). This fact is the essential characteristic for Case I. It can now be shown that, when $x_1 = x'$, I ceases to be a minimum. Let $P = (x'', y'')$ be an arbitrary point of the envelope lying in the neighborhood of

$B = B' = (x', y')$ and let $x'' < x'$. Then the value of I taken from A to B along C is the same as its value taken first along the extremal AP and then along the envelope from P to B. The proof of this theorem is readily deduced from the analysis by which WEIERSTRASS's sufficient condition is established.* Thus the function $f(s)$ (loc. cit., §3) formed for the curve $\bar{C}$ consisting of APB is easily shown to be constant throughout the entire interval $0 \leqq s \leqq l$. If we employ HILBERT's method (loc. cit., §5), the proof is still simpler. Let $\bar{C}$ be a curve connecting A with B' and lying within the region bounded by C and the curve APB'. Then it follows, as in §2 of this article, that HILBERT's integral J has the same value for $\bar{C}$ as for C. When $\bar{C}$ goes over continuously into APB', the integral J goes over continuously into this integral taken along APB', and the proof is complete.

Thus far it has only been shown that the integral I loses the minimum property, inasmuch as there are curves other than C lying in the neighborhood of C for which I has the same value as for C. By making use of the relation †

$$\phi'_\gamma(x', \gamma_0) \neq 0,$$

we can show that the envelope PB' is not an extremal, and hence that there exist curves in the neighborhood of C for which I has a smaller value than for C. In fact, for the envelope,

$$\frac{dy}{dx} = \phi'(x, \gamma), \qquad \frac{d^2y}{dx^2} = \phi''(x, \gamma) - \frac{\{\phi'_\gamma(x, \gamma)\}^2}{\phi_{\gamma\gamma}(x, \gamma)} \neq \phi''(x, \gamma),$$

and so the envelope does not satisfy LAGRANGE's equation (1).

In Case II no envelope in the proper sense of the term exists. All the extremals $y = \phi(x, \gamma)$ for which $\gamma_0 - \delta < \gamma < \gamma_0 + \delta$ go through the point (x', y'). The value of I taken along each of these extremals is the same, as can be shown by reasoning similar to that employed in Case I, and hence I ceases to be a

* Cf. Annals of Mathematics, loc. cit., §§ 3, 5.

† Cf. KNESER's memoir, p. 40, formula (32).

minimum. If the function $\phi(x, \gamma)$ satisfies all of the conditions requisite for a field about the arc AB' of C with the one exception that through the point (x', y') more than one extremal passes, there will be no curve $\overline{C}$ in the neighborhood of C for which I has a smaller value than for C. The great circles of a sphere, regarded as shortest lines on the sphere, are typical for this case, A and B' being two points situated diametrically opposite each other.

§ 4. *The Revision of Kneser's Theorem.*

In Case III an envelope may exist which has a singular point (e. g., cusp) at B' and *does not enter the interval* (x_0, x'). This case KNESER appears to have overlooked. * All of the conditions laid down for a field in § 2 may be fulfilled, so that the integral I will still be a minimum, as is shown by the example of the following section. On the other hand, there are cases in which the conditions of Case III are fulfilled, but in which the envelope enters the region (x_0, x') and the reasoning employed in Case I to show that I ceases to be a minimum can be repeated.

Let us, then, examine the possibilities which Case III presents with some detail. Denote the set of points (x, y) lying in the neighborhood of B', for which

$$\phi_\gamma(x, \gamma) = 0$$

by (E). The point B' is a member of (E). We will distinguish two cases.

(a) The set (E) contains a point (x, y) for which $x < x'$.

(b) The set (E) contains no such point.

In case (a) the integral I will cease to be a minimum whenever the points of (E) — or some of these points — for which $x \leqq x'$ lie on a curve E which has a continuously turning tangent and goes through B'; which, furthermore, at each point of E in the neighborhood of B', is tangent to the extremal $y = \phi(x, \gamma)$ that passes through that point; and when lastly $\phi_\gamma(x, \gamma) \neq 0$ at all points on the segment of each line parallel to the y-axis that lies between E and C. For, the proof given in Case I applies here.

A sufficient condition for the existence of case (a) is that in the neighborhood of the point $\gamma = \gamma_0$, the derivatives $\phi_{\gamma^k}(x', \gamma)$, $k = 1, 2, \cdots, n$, exist; that $\phi_{\gamma^k}(x', \gamma_0)$ vanishes when $k < n$, but does not vanish when $k = n$; and that n is even. Case (a) may, however, exist when n is odd.

For, let $\phi_{\gamma^n}(x', \gamma_0) > 0$; n even. Then the function $y = \phi(x', \gamma)$ is a minimum when $\gamma = \gamma_0$, and hence in the neighborhood of the point $\gamma = \gamma_0$, $\phi(x', \gamma) > \phi(x', \gamma_0)$ both for values of γ larger and for values of γ smaller than γ_0. Now at any point $(\bar{x}, \bar{y})$ of C intermediate between A and B', $\phi_\gamma(\bar{x}, \gamma_0) \neq 0$, and if $\phi_\gamma(\bar{x}, \gamma_0) > 0$, the curves $y = \phi(x, \gamma)$ for which

* Cf., however, below, § 6.

$\gamma_0 - \eta < \gamma < \gamma_0$ all lie below C at the point $x = \bar{x}$ and hence cut C between $\bar{x}$ and x'. The equation

$$\phi(x, \gamma) - \phi(x, \gamma_0) = (\gamma - \gamma_0)\phi'_\gamma\big(x, \gamma_0 + \theta(\gamma - \gamma_0)\big) = 0$$

has a solution $x = X$, $\bar{x} < X < x'$, and $\big(X, \gamma_0 + \Theta(\gamma - \gamma_0)\big)$ is a point of (E).

Thus for example if $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of (x', γ_0) and if

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \quad (k = 1, 2, \cdots, \quad n-1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

n being even, we shall certainly have case (a). But more than this: with the present restrictions on the function $\phi(x, \gamma)$ we can infer that all the conditions named above are fulfilled which suffice that I cease to be a minimum. In fact, the equation

$$0 = \phi_\gamma(x, \gamma)$$

now defines a set of points (E) that may be grouped together so as to constitute a finite number of curves passing through B' and not intersecting one another elsewhere in the neighborhood of B'. Through that part of the neighborhood of B' for which $x < x'$ at least one of these curves will pass, and if we denote by E that one of these curves that lies next to C (on either side), then E will satisfy all of the above requirements. For, it is possible to write $\phi_\gamma(x, \gamma)$ in the form

$$\phi_\gamma(x, \gamma) = \bar{\phi}\,\Phi,$$

where Φ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and does not vanish there; and where $\bar{\phi}$ is of the form:

$$\bar{\phi} = \gamma^n + A_1\gamma^{n-1} + \cdots + A_n,$$

the coefficients A_i being analytic functions of x in the neighborhood of the point $x = x'$.* The irreducible factors † of this polynomial, set equal to zero, yield the curves which constitute the set (E). In particular, let $\bar{\phi}$ be the factor which yields the curve lying next to C. This curve satisfies all the conditions imposed on E. For, since $\bar{\phi}$ is irreducible, $\bar{\phi}_\gamma$ does not vanish simultaneously with $\bar{\phi}$ in the neighborhood of (x', γ_0) when $x < x'$, and hence

$$\frac{d\gamma}{dx} = -\frac{\bar{\phi}_x}{\bar{\phi}_\gamma}$$

exists and is finite at all such points. The slope of the curve is given by the equation

* This is a fundamental theorem of WEIERSTRASS's; Werke, vol. 2, p. 135.

† "Irreducible" in the sense in which I have used this term in reporting WEIERSTRASS's theorem, Encyclopädie der Mathematischen Wissenschaften, II., B. 1.

$$\frac{dy}{dx} = \phi' + \phi_\gamma \frac{d\gamma}{dx} = \phi',$$

where $x < x'$, and it is readily shown from the continuity of $\phi'(x, \gamma)$ that at B' the slope of E is given by the same formula.

If all of the foregoing conditions hold except the one that n be *even*, and if now n is odd, it would appear that we may still have case (a) and hence the existence of an envelope E with the above properties. For, the function

$$\phi(x, \gamma) = (x+1)(x-\gamma)^3, \quad x_0 = -1, \quad x' = 0, \quad \gamma_0 = 0,$$

for example, satisfies all the conditions imposed on the function γ except, possibly, that of being an extremal. It seems probable that there are families of extremals for which this example is typical.

In case (b) a field will exist and I will continue to be a minimum provided that through B' and each point of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes. *

This will always be the case when it is the case for the points of the line $x = x'$ that lie in the neighborhood of B'. For then γ will be a single valued function of y along this line, continuous, monotonic and never constant, since the inverse function, $y = \phi(x', \gamma)$ is continuous and single valued.

Consider the region bounded by the quadrilateral whose sides are

$$y = \phi(x, \gamma_0 + \epsilon), \quad y = \phi(x, \gamma_0 - \epsilon), \quad x = x', x = x' - \epsilon > x_0,$$

where ϵ is so chosen (1) that the points $y' = \phi(x', \gamma_0 \pm \epsilon)$ are points of the above neighborhood of y' and (2) that $\phi_\gamma(x, \gamma)$ does not vanish within the quadrilateral. Let $(\bar{x}, \bar{y})$ be any point lying within the quadrilateral. Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits one and only one solution $\gamma_0 - \epsilon < \gamma < \gamma_0 + \epsilon$. For

$$\phi(\bar{x}, \gamma_0 - \epsilon) < \bar{y} < \phi(\bar{x}, \gamma_0 + \epsilon)$$

and $\phi(\bar{x}, \gamma)$ is a continuous function of γ in the interval $\gamma_0 - \epsilon \leqq \gamma \leqq \gamma_0 + \epsilon$. Hence $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ for at least one value of γ in the interval. Since $\phi_\gamma(\bar{x}, \gamma)$ does not vanish in the interval, it follows from ROLLE'S theorem that the equation $\phi(\bar{x}, \gamma) = \bar{y}$ has only one root in the interval. Thus the sufficiency of the above condition is established.

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and we have case (b), then the integral will always be a minimum. For, only a finite number of the derivatives $\phi_{\gamma^k}(x', \gamma_0)$ can vanish. The first one

* More precisely: provided that, if $(\bar{x}, \bar{y})$ is the point B' or any point of its neighborhood, for which $x < x'$, the equation $\bar{y} = \phi(\bar{x}, \gamma)$ has one and only one root γ lying in the neighborhood of γ_0.

that does not vanish cannot be of even order, otherwise we should have case (a). The order being odd, it is readily seen that the equation $y = \phi(x', \gamma)$ admits, for each value y in the neighborhood of y', one and only one root γ in the neighborhood of γ_0.

The foregoing results may be summarized in the following theorems, which may be regarded as a correction and extension of KNESER's theorem.

THEOREM A. *The integral*

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

ceases to be a minimum when x_1 is a conjugate point, $x_1 = x'$, of x_0 and the conditions of Case I *or* II *are fulfilled.*

This is KNESER's theorem accurately enunciated.

THEOREM B. *In Case* III, *the integral ceases to be a minimum when we have case (a), i. e., when some of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie in the interval (x_0, x'), and when, besides, such points, or a part of them, can be so grouped as to form the arc of an envelope E entering the region (x_0, x') and, together with the extremal C and the line $x = x' - \epsilon$, bounding a region within which $\phi_\gamma(x, \gamma) \neq 0$.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and if we have case (a), then the further conditions of this theorem will be fulfilled. Here, a sufficient, but apparently not a necessary, condition that we have case (a) is that

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \quad (k = 1, 2, \cdots, n-1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

where n' is even.

THEOREM C. *In Case* III, *the integral continues to be a minimum when we have case (b), i. e., when none of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie within the interval (x_0, x'); and when, besides, through B' and each point (x, y) of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and if we have case (b), then the further conditions of this theorem will be fulfilled.

These theorems exhaust completely the case that $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), the case to which KNESER restricts his investigation.

§ 5. *The Geodesics on an Ellipsoid of Revolution.*

The geodesics on an oblate ellipsoid of revolution furnish an example of an integral I which does not always cease to be a minimum when x_1 is a conjugate point of x_0.

The form of the envelope of the geodesics drawn from a fixed point on an ellipsoid of revolution was given by JACOBI* and the envelope was investigated analytically by VON BRAUNMÜHL.†

Let the equation of a surface of revolution be written:

$$y = f(r), \quad \text{where} \quad r^2 = x^2 + z^2,$$

and let us take as the coördinates of a point on the surface r and the angle ω that the plane of the meridian through the point makes with a fixed plane through the axis of revolution. The length of the arc of a curve joining two points (r_0, ω_0) and (r_1, ω_1) is given by the formula

$$\int_{\omega_0}^{\omega_1} \sqrt{r^2 + \left(1 + f'(r)^2\right) r'^2}\, d\omega.$$

This integral is to be made a minimum. Since the integral does not contain the independent variable explicitly, LAGRANGE'S equation (1) admits a first integral and the differential equation of the extremals (geodesics) can be written in the form:‡

$$r^2(r^2 - v^2)\, d\omega^2 = v^2(1 + f'(r)^2)\, dr^2, \tag{2}$$

where v is the constant of integration.

The surface here to be considered is the oblate ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = ca, \quad c < 1, \tag{3}$$

and the point A through which all the extremals pass is to be a point of the equator, for which ω shall vanish. The extremal C is to be the equator, $r = a$. We will choose ω as the independent variable and replace it by x to conform to the earlier notation, and y as dependent variable. The function $F_{y'y'}$ is seen to be positive for all values $-b < y < b$ and $-\infty < y' < \infty$; it does not contain x explicitly. Equation (2) becomes the following:

$$(b^2 - y^2)^2 \{c^2(a^2 - v^2) - y^2\} dx^2 = c^2 v^2 \{c^2 b^2 + (1 - c^2) y^2\} dy^2.$$

To integrate this equation set

$$\gamma = \pm c\sqrt{a^2 - v^2}, \qquad y = \gamma \sin \theta.$$

**Vorlesungen über Dynamik*, 6th lecture, Werke, Supplementband, p. 47, where the form of the envelope is indicated by a figure.

†Mathematische Annalen, vol. 14 (1879), p. 557. Cf. also the models made by VON BRAUNMÜHL in the Brill (now Schilling) collection, Katalog mathematischer Modelle, ser. 5, XVIII, a, b = Specialkatalog, 104, 106 = Deutsche Mathematiker-Vereinigung, Katalog mathematische Modelle, München, 1892, 2. Teil, II. Abteilung, 214.

In a second paper by the same author (Mathematische Annalen, vol. 20 (1882), p. 557) the geodesics on an ellipsoid with three distinct axes are treated and a number of bibliographical references are given.

‡KNESER, *Lehrbuch der Variationsrechnung*, p. 24.

Then x is given as a function of θ by a quadrature and we have the formulas:

$$(4) \qquad \begin{cases} y = \gamma \sin\theta, \\ x = \sqrt{b^2-\gamma^2}\int_0^\theta \dfrac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta, \end{cases}$$

where all the radicals are taken positive and $-b<\gamma<b$. For any one of these values of γ, x is a single valued continuous function of θ that increases when θ increases, and hence the inverse function, $\theta(x)$, is also single valued and increases as x increases. Equations (4) thus define y as a single valued function of (x, γ) for all values of (x, γ) in the domain:

$$-\infty<x<\infty, \quad -b<\gamma<b,$$

and it follows from DINI'S theorem (cf. p. 170) that this function, $y=\phi(x,\gamma)$ has continuous partial derivatives at each point of this region.

The course of the geodetics on the ellipsoid is shown in VON BRAUNMÜHL'S first paper (loc. cit.), Figs. 1–3, and on the models. It is suggested geometrically by Fig. 3 and the corresponding model that either cusp of the envelope lying on the equator is a point x' conjugate to A for the portion of the equator C included between A and x'; that this portion of C is embedded in a family of extremals having an envelope passing through x'; and that this envelope does not enter the region Ax'. Thus it would appear that this portion of C is surrounded by a field and hence that for it the integral is a minimum. It remains to supply the corresponding analytic proofs.

In the case of the prolate ellipsoid, Fig. 1 of VON BRAUNMÜHL'S paper, it is evidently the cusps on the meridian through A that yield the desired example.

For a given value of γ lying between $-b$ and b, the curve $y=\phi(x,\gamma)$ is an extremal and its course on the ellipsoid is easily traced. Since θ always increases with x, the curve lies within the zone of the ellipsoid bounded by the planes $y=\pm\gamma$. It is a periodic curve and the value of x corresponding to a quarter period is given by the formula

$$\lambda = \sqrt{b^2-\gamma^2}\int_0^{\frac{\pi}{2}} \frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta.$$

For this value of x, y is a maximum, $=\gamma$. The curve crosses the equator, $y=0$, when $\theta=\pi$, $x=2\lambda$. Let γ converge toward 0; then 2λ approaches πc as its limit and, as will appear in the course of the subsequent work,

$$x' = \pi c$$

is the conjugate point to A $(x=0)$ on the equator, C.

For all values of x that are positive and less than πc, $\phi_\gamma(x,\gamma)\neq 0$ when $\gamma=\gamma_0=0$. In act, x is then equal to $c\theta$, so that $0<\theta<\pi$; and since

$$\phi_\gamma(x,\gamma)=\frac{\partial y}{\partial\gamma}=\sin\theta+\gamma\cos\theta\frac{\partial\theta}{\partial\gamma};\tag{5}$$

it follows that $\phi_\gamma(x,\gamma_0)=\sin\theta>0$.

At the point $x=\pi c$, $\gamma=0$, the function $\phi_\gamma(x',\gamma_0)$ vanishes; and the same is true of the function

$$\phi_{\gamma\gamma}(x,\gamma)=\frac{\partial^2 y}{\partial\gamma^2}=2\cos\theta\frac{\partial\theta}{\partial\gamma}-\gamma\sin\theta\left(\frac{\partial\theta}{\partial\gamma}\right)^2+\gamma\cos\theta\frac{\partial^2\theta}{\partial\gamma^2}.$$

For, $\partial\theta/\partial\gamma$ is given by the formula:

$$\begin{aligned}0=&\sqrt{b^2-\gamma^2}\,\frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}\,\partial\theta}{b^2-\gamma^2\sin^2\theta\qquad\partial\gamma}\\&-\frac{\gamma}{\sqrt{b^2-\gamma^2}}\int_0^\theta\frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta\\&+\sqrt{b^2-\gamma^2}\int_0^\theta\left\{\frac{(1-c^2)\gamma\sin^2\theta}{(b^2-\gamma^2\sin^2\theta)\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}\right.\\&\qquad\left.+\frac{2\gamma\sin^2\theta\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{(b^2-\gamma^2\sin^2\theta)^2}\right\}d\theta\end{aligned}\tag{6}$$

and hence $\partial\theta/\partial\gamma$ vanishes when $\gamma=0$. The function $\phi_{\gamma\gamma}(x',\gamma)$ does not, however, vanish identically at the point $\gamma=0$. In fact, $\phi_{\gamma^3}(x',\gamma_0)\neq 0$. For

$$\phi_{\gamma^3}(x,\gamma)=2\cos\theta\frac{\partial^2\theta}{\partial\gamma^2}+\rho,$$

where ρ is composed of terms that vanish at the point (x',γ_0), and it is readily shown by direct computation that $\partial^2\theta/\partial\gamma^2$ does not vanish at this point.

We proceed now to show that the further conditions of Theorem C, § 4, are fulfilled, and hence that the integral taken along C from A to B' is a minimum. It remains to establish two things: (α) that through each point (x,y) of the neighborhood of A, for which $x>0$, one and only one extremal $\phi(x,\gamma)$ passes, and that at each of these points $\phi_\gamma(x,\gamma)\neq 0$; and that, furthermore, $\phi'(x,\gamma)$ remains finite in that part of this neighborhood lying between any two curves, y_1 and y_2, of the class considered in § 1; (β) that at the point B' we have case (b) (§ 4), for $\phi(x,\gamma)$ is an analytic function of (x,γ) in the neighborhood of B', and hence the existence of case (b) suffices.

The proof of (β) is direct. At all points of the region

$$x'-\epsilon<x<x',\quad -\kappa<\gamma<\kappa,$$

where ϵ, κ are sufficiently small positive constants, $\phi_\gamma(x,\gamma)>0$. For, an approximate evaluation of x by (4) in terms of γ near 0 and θ near π shows that

when x and γ lie in the above region, θ is less than π. Similarly it appears from (6) that for such values of θ and γ, $\gamma\,\partial\theta/\partial\gamma \leqq 0$. Hence the first term in the right hand member of (5) is always positive and the second is never negative.

Finally, to establish (a), we show (a) that through each point (x, y) of the neighborhood of A for which $x > 0$ at least one extremal $y = \phi(x, \gamma)$ passes; (b) that for all such points $0 < \theta < \delta$, where δ can be made arbitrarily small by choosing the above neighborhood suitably; (c) that $\gamma\,\partial\theta/\partial\gamma \geqq 0$ when $0 < \theta < \delta$, whence it follows by the aid of (5) that $\phi_\gamma(x, \gamma) > 0$; (d) that $\phi'(x, \gamma)$ remains finite between any two extremals $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$.

ad (a). Let ϵ, η be two positive quantities, the first arbitrarily small, the second, any quantity between 0 and 1, e. g., $\eta = \frac{1}{2}$. Then through each of the points

$$y = Y = \eta b \sin\epsilon, \quad 0 < x \leqq \epsilon',$$

where ϵ' denotes the value of x corresponding to the values $\gamma = \eta b$, $\theta = \epsilon$, there passes at least one extremal $\phi(x, \gamma)$. For, the equation

$$\gamma \sin\theta = \eta b \sin\epsilon$$

defines θ as a single valued continuous function of γ that decreases from the value ϵ to the value $0 < \sin^{-1}(\eta \sin\epsilon) < \epsilon$ when γ increases from the value ηb to b. Substitute this function $\theta(\gamma)$ in the formula for x, (4). Then x becomes a continuous function of γ in the interval $\eta b \leqq \gamma \leqq b$, assuming at the extremities of the interval respectively the values ϵ' and 0, and hence assuming each intermediate value at least once.

Thus the equation

$$Y = \phi(\bar{x}, \gamma),$$

where $\bar{x}$ is chosen arbitrarily between 0 and ϵ', admits at least one solution $\gamma = \Gamma$, where $\eta b < \Gamma < b$. Let $(\bar{x}, \bar{y})$ be any point of the region

$$0 < \bar{x} < \epsilon', \quad 0 \leqq \bar{y} < Y.$$

Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits at least one solution, $\gamma = \bar{\gamma}$, where

$$0 \leqq \bar{\gamma} < b.$$

For

$$\phi(\bar{x}, 0) \leqq \bar{y} < \phi(\bar{x}, \Gamma),$$

and the continuous function $\phi(\bar{x}, \gamma)$ must assume the value $\bar{y}$ at least once when γ passes from 0 to Γ. To show that $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ only once,

it is sufficient to show that $\phi_\gamma(\bar{x}, \gamma) > 0$, when $0 < \gamma < b$ and this relation follows from (b) and (c).

ad (b). At any point $(\bar{x}, \bar{y})$ at which $\gamma \geqq \eta b$,

$$\theta \leqq \epsilon, \quad \text{since} \quad \eta b \sin\theta \leqq \gamma \sin\theta = \bar{y} < Y = \eta b \sin\epsilon.$$

On the other hand, when $\gamma < \eta b$, it appears from (4) that

$$\bar{x} > \sqrt{b^2 - \eta^2 b^2} \int_0^{\theta} \frac{cb}{b^2 - \eta^2 b^2}\, d\theta = \frac{c}{\sqrt{1-\eta^2}}\theta,$$

or

$$\theta < c^{-1}\sqrt{1-\eta^2}\,\bar{x} < c^{-1}\sqrt{1-\eta^2}\,\epsilon'.$$

Hence δ may be taken equal to the larger one of the quantities ϵ, $c^{-1}\sqrt{1-\eta^2}\,\epsilon'$.

ad (c). From (6) it is clear that for all values of $-b < \gamma < b$ and for all values of $0 < \theta < \delta$, where δ is chosen sufficiently small, the sign of $\gamma\,\partial\theta/\partial\gamma$ is dominated by the sign of the first integral, and hence is positive.

From (5) it now follows that, when ϵ is chosen sufficiently small, $\phi_\gamma(x, \gamma) > 0$ at all points (x, y) of the neighborhood of A for which $x > 0$, $y \geqq 0$. Similar reasoning applies to the region $x > 0$, $y < 0$.

ad (d). The value of $\phi'(x, \gamma)$ is given by the formula

$$\phi'(x, \gamma) = \frac{dy}{dx} = \frac{\gamma}{\sqrt{b^2 - \gamma^2}} \frac{\cos\theta(b^2 - \gamma^2\sin^2\theta)}{\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}},$$

from which it follows that $|\phi'(x, \gamma)| < b^2 c^{-1}(b^2 - \gamma^2)^{-\frac{1}{2}}$. On the other hand, two extremals, $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$ can always be so chosen as to include between them that part of the neighborhood in question which lies between the curves y_1 and y_2. Hence the finiteness of $\phi'(x, \gamma)$ follows without difficulty.

§6. *The Integral* $I = \int_{t_0}^{t_1} F[x, y, x', y']\,dt$.*

In the foregoing, the value of the integral taken along C has been compared

* This section was added 7 March. At the time that I sent the MS. to the TRANSACTIONS, I had not observed that KNESER, in his treatise, § 25, notes the case in which the theorem cited in § 1 of this paper fails as an "exceptional case" (p. 95, l. 4). He recognizes that it is conceivable that the envelope may have a cusp, without however showing that this is ever actually the case. Near the close of the section he says: "Eine Ausnahme findet nur statt, wenn die stets vorhandene Enveloppe der Curven des Feldes im Brennpunkte einen Rückkehrpunkt von besonderer Art besitzt." In what the specialization of the cusp consists is not explained, and, in fact, it is not even clear whether the "Ausnahme" is considered to be an exception to the *proof* or an exception to the *theorem*.

with its value taken along only such curves of the neighborhood of C as are cut by a parallel to the y-axis in at most one point. In the case of the geodesic AB' on the ellipsoid it is readily seen that it continues to be a minimum even when compared with any continuous curve having a continuously turning tangent, which can be drawn from A to B' in the neighborhood of C. (The neighborhood of C shall now include the whole neighborhood of the points A and B'.) For, to begin with, the integral being written in the Weierstrassian form with a parameter t:

$$I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt, \tag{7}$$

it is easily seen that the field may be taken as the neighborhood of C with the exception of that part of this region that is cut out by the envelope. The Weierstrassian proof for the existence of a minimum will apply to any curve connecting A and B' and lying in this field.

Next, consider the family of geodesics through A that lie in the neighborhood of C, and lay off on each of these a length measured from A equal to the length of the geodesic AB'. The locus of the points thus determined is readily seen to be a curve Γ passing through B' and not tangent to C.

Now consider *any* curve lying in the neighborhood of C and connecting A and B'. If this curve meets Γ only at the point B', it lies wholly in the field, and hence is longer than the geodesic AB'. If it meets Γ before it reaches B', then this part of it is at least as long as AB', and hence in this case, too, its total length is greater than that of the arc AB' of the geodesic C.*

It is possible to generalize from this example as follows. Let A, B' be two conjugate points on the extremal C of the general integral of the form (7). If the extremals of the field are given by the formulas†

$$x = \xi(t, a), \qquad y = \eta(t, a),$$

then, to the former function $\phi_\gamma(x, \gamma)$ corresponds now the determinant

$$\Delta = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.$$

We assume that Δ vanishes only at A and at points in the neighborhood of B'. If, furthermore, $\xi(t, a)$, $\eta(t, a)$ are both analytic in the neighborhood of B', the equation

$$\Delta = 0$$

will define one or more curves going through B' and tangent to C there.

* It may furthermore be shown that the arc AB' is shorter than any other curve that can be drawn on the ellipsoid connecting A and B'. For, the curve Γ is a closed curve that does not cut itself, and the field may be so chosen as to include the whole interior of this curve.

† KNESER, *Variationsrechnung*, chap. 3.

Divide the neighborhood S of C into two pieces, S_1 (containing A) and S_2, by a curve through B' cutting C normally. Then to cases (a) and (b) of § 4 correspond here the cases (a) that there are interior points of S_1 at which $\Delta = 0$ (we will also tabulate here the case that $\Delta = 0$ at B', but nowhere else in the neighborhood of B'); (b) that there are no such points, but that there are points within S_2 in the neighborhood of B' at which $\Delta = 0$. In case (a) the integral I will cease to be a minimum. In case (b) the integral will certainly continue to be a minimum when compared with the value that it attains when taken along any curve of the neighborhood of C that meets the locus $\Delta = 0$ only at B', for such a curve will lie in the field; and it will continue to be a minimum even when a curve of the kind just excluded is considered, provided that the integrand

$$F(x_1, y_1, \cos\phi, \sin\phi) > 0,$$

where the coördinates of B' are written as (x_1, y_1) and ϕ has any value. For, the locus of points so taken on the extremals which lie in the neighborhood of C that the value of I along each of these extremals is the same as the value of I along C from A to B' is a curve Γ passing through B' and not tangent to C, and the points at which $\Delta = 0$ lie on a curve or curves tangent to C at B'. The remainder of the proof follows the same lines as the proof of the example above considered.

ON THE GEOMETRY OF PLANES IN A PARABOLIC SPACE OF FOUR DIMENSIONS*

BY

IRVING STRINGHAM

Literature.

Of the literature of the geometry of hyperspace that has accumulated in recent years the following papers are cited as having points of contact with the ideas here set forth:

CLIFFORD: *Preliminary Sketch of Biquaternions* in Proceedings of the London Mathematical Society, vol. 4 (1873), pp. 381–395. CLIFFORD'S theory of parallels in elliptic space is identical with the theory of isoclinal systems of planes in four-dimensional space; namely, planes that pass through a fixed point and make equal dihedral angles with any transversal plane through the same point. (See §§ 30–32 of this paper.)

CHARLES S. PEIRCE: Reprint of the *Linear Associative Algebra* of BENJAMIN PEIRCE in the American Journal of Mathematics, vol. 4 (1881). In the foot-note of page 132 attention is called to the fact that in four-dimensional space two planes may be so related to one another that every straight line in the one is perpendicular to every straight line in the other. (See § 28 (3) of this paper.)

I. STRINGHAM: (1) *On a Geometrical Interpretation of the Linear Bilateral Quaternion Equation;* (2) *On the Rotation of a Rigid System in Space of Four Dimensions;* (3) *On the Measure of Inclination of two Planes in Space of Four Dimensions.* Papers presented to Section A of the American Association for the Advancement of Science, the first two at the Philadelphia meeting of 1884, the third at the Cleveland meeting of 1888. Abstracts printed in Proceedings of the Association, 1884, pp. 54–56, and privately, 1888. These papers form the nucleus of the present investigation.

A. BUCHHEIM: *A Memoir on Biquaternions*, in the American Journal

* Presented to the Society (Chicago) December 27, 1900. Received for publication December 7, 1900.

of Mathematics, vol. 7 (1885). The geometrical part is devoted to the geometry of non-Euclidean space. The theory of parallels is discussed on pp. 301–306, 316–325.

W. KILLING: *Die nicht-Euklidischen Raumformen*, Leipzig, 1885. Section 8, pp. 148–160, discusses, under the title: "Die gegenseitige Lage zweier Ebene," the relations to one another of μ- and ν-dimensional planes in an n-dimensional space.

CAYLEY: *On the quaternion Equation* $qQ - Qq' = 0$, and *On the Matrical Equation* $qQ - Qq' = 0$; Messenger of Mathematics, vol. 14 (1885), pp. 108–112 and 300–304, or Mathematical Papers, vol. 12, pp. 300–304 and 311–313. The first paper discusses the character of the roots of the equation $qQ - Qq' = 0$; the second interprets q, q', Q as matrices.

KLEIN: *Vorlesungen über nicht-Euklidische Geometrie*, II., 1890 (Zweiter Abdruck, 1893). On pages 120–124 the author identifies quaternion multiplications with orthogonal substitutions in four variables. In particular the products $p \cdot q$ and $q \cdot p$ are called respectively "eine Schiebung erster Art" and "eine Schiebung zweiter Art"; these are the parallel and contra-parallel translations of Professor HATHAWAY's paper number (1), cited below. CLIFFORD's theory of parallels in elliptic space is explained in the Vorlesungen on pages 228–237. KLEIN's theory of Schiebungen was first presented to the members of his Seminar in January and February, 1880.

G. VERONESE: *Fondamenti di Geometria*, Padua, 1891, Part II, Book I, p. 455 et sq. Chapter 1 discusses elementary theorems in the geometry of four-dimensional space, some of which deal with the perpendicularity, parallelism, and intersections of planes.

M. BRÜCKNER: *Die Elemente der vierdimensionalen Geometrie mit besonderer Berücksichtigung der Polytrope;* Jahresber. d. Ver. f. Naturk., Zwickau, 1893. I have not been able to consult a copy of this paper, but SCHLEGEL, in Fortschritte der Mathematik, vol. 25, p. 1028, says: "Die ··· Arbeit giebt eine auf gründlicher Litteraturkenntnis beruhende und durch grosse Klarheit in der Darstellung sich empfehlende Zusammenstellung der in der elementaren vierdimensionalen Geometrie erzielten Resultate, und füllt dadurch in erwünschter Weise eine Lücke in der deutschen Litteratur aus."

P. CASSINI: *Sulla geometria pura Euclidiana ad n dimensioni;* Atti del Reale Instituto Veneto (7), vol. 5 (1894), p. 820 et sq., discusses metrical geometry in four-dimensional space; but see Fortschritte der Mathematik, vol. 25, p. 1035.

A. S. HATHAWAY: (1) *Quaternions as Numbers of Four-Dimensional Space*, in Bulletin of the American Mathematical Society, vol. 4 (1897), pp. 54–57; (2) *Alternate Processes*, in Proceedings of the Indiana Academy of Sciences, Indianapolis, 1897, pp. 1–10; (3) *Linear*

Transformations in Four-Dimensional Space, in Bulletin of the American Mathematical Society, vol. 5 (1898), pp. 93–94. The fundamental ideas and formulæ of these papers are the same as those that were used in my papers of 1884 and 1888. Specifically Professor HATHAWAY exemplifies the utility of the quaternion analysis for the following purposes: (1) To interpret CLIFFORD'S theory of parallels in elliptic space (KLEIN'S Schiebungen), there stated in terms of great circular arcs on the hypersphere of four-dimensional space (see this paper, §§ 30–32); (2) To determine certain angles, areas, and volumes in four-dimensional space; (3) To formulate the theory of certain four-dimensional space transformations, in particular rotations (cf. my paper numbered (2) above).

A. N. WHITEHEAD: *Universal Algebra*, vol. 1 (1898). At pp. 405–406, 409 the properties of parallels in elliptic space are explained by means of the Calculus of Extension (Ausdehnungslehre).

§§ 1–12. SOME FUNDAMENTAL CONSIDERATIONS.

1. *The Quaternionic Manifold.* The four-dimensional space here considered is a point-manifold whose point-elements are uniquely determined by the sets of real variable numbers w, x, y, z, regarded as rectangular coördinates. To real numbers shall correspond always and only real points.

The space may be defined as the domain of the continuous translational and rotational transformation groups expressed in terms of the coördinates. The translations are the linear transformations of the form:

$$w + g, \quad x + a, \quad y + b, \quad z + c;$$

and the rotations are the orthogonal transformations of the type:

$$g_i w + a_i x + b_i y + c_i z \qquad (i = 1, 2, 3, 4).$$

In the quaternion analysis these are respectively additions and multiplications, and the coördinates of the transformed point are, in every case, the resultant coefficients of the fundamental units $1, i, j, k$. The quaternion terms, or factors, being

$$q = g + ai + bj + ck, \quad p = w + xi + yj + zk,$$

translations are represented by the sum

$$p + q = (w + g) + (x + a)i + (y + b)j + (z + c)k,$$

rotations by the products: pq, qp, qpq'^{-1}, etc., provided (in the latter case) the condition $Tq = Tq' =$ a constant be assigned. Thus, in particular,

$$qp = W + Xi + Yj + Zk,$$

where

$$W = gw - ax + by - cz,$$
$$X = aw + gx - cy + bz,$$
$$Y = bw + cx + gy - az,$$
$$Z = cw - bx + ay + gz,$$

and in these equations all the conditions of orthogonality are satisfied if

$$g^2 + a^2 + b^2 + c^2 = 1.$$

Such multiplications obviously constitute an orthogonal group, that is a group of rotations in the quaternionic manifold, rotations having here the same meaning as when expressed in terms of the Cartesian analysis.*

Now it is well known that the continuous translational and rotational transformation groups (expressed in the Cartesian form) constitute the totality of the possible real movements, without distortion, in a parabolic space defined by the variables w, x, y, z; and the quaternion operations do actually reproduce such transformation groups, are therefore competent to interpret the geometry of a parabolic space of four dimensions. But do quaternion additions and multiplications suffice to produce *all* the movements (without distortion) of such a space?

In order to answer this question I assume that a rigid body is fixed by four points that have no special relation to one another, e. g., do not lie in a space of two dimensions, and I then show that, by quaternion additions and multiplications, any one set of four points o_1, a_1, b_1, c_1 can be moved into any second set o', a', b', c' congruent to the first, but otherwise arbitrarily placed. The letters here used denote quaternions except where statement to the contrary is made.

Place the origin at the point defined by o', so that o', regarded as a quaternion, has a zero tensor.

By a first operation of addition applied to each of the points o_1, a_1, b_1, c_1, the additive term being $o' - o_1$, we transform (o_1, a_1, b_1, c_1) into (o', a, b, c) where

$$a, b, c = -o_1 + a_1, -o_1 + b_1, -o_1 + c_1,$$

and the application of the proper test shows that (o', a, b, c) is actually congruent to (o_1, a_1, b_1, c_1)

By a second operation of multiplication (o', a, b, c) can now be transformed into (o', a', b', c'). In fact, the origin being still at o', if it be defined in general that

$$r(a, b, \cdots)r' = (rar', rbr', \cdots),$$

* Cf. KLEIN: loc. cit. and *Vorlesungen über das Ikosaeder* (1884), pp. 35, 36; also CAYLEY: *On Certain Results Relating to Quaternions* in Philosophical Magazine, vol. 26 (1845), pp. 141–145, or Mathematical Papers, vol. 1, pp. 123–126.

it is easily verified that

$$q(o', a, b, c)a^{-1}q^{-1}a' = (o', a', b', c'),$$

where

$$q = \beta'(\gamma' - \gamma) + (\gamma' - \gamma)\beta$$

and

$$\beta, \gamma, \beta', \gamma' = \mathrm{V}ba^{-1}, \mathrm{V}ca^{-1}, \mathrm{V}b'a'^{-1}, \mathrm{V}c'a'^{-1};$$

and again, the application of the proper test shows that (o', a', b', c') is actually congruent to (o', a, b, c).

By means of appropriate quaternion operations we may therefore transport without distortion the elements and configurations of our quaternionic space from any given position to any other.

The system of measurement is parabolic. But one should not forget that the quaternion analysis augmented into a calculus of biquaternions (CLIFFORD, BUCHHEIM), or of octonions (MCAULAY), may suffice to interpret both the elliptic and the hyperbolic forms of a four-dimensional space.

In concluding this preliminary discussion it is pertinent to remark that *the quaternion theory makes its interpretations in strict analogy with the vector interpretations of a parabolic space of three dimensions* and, so far as they are known, takes them for granted. The quaternionic four-dimensional space is therefore the analogue of that three-dimensional space which allows itself to be explained by the Hamiltonian vector analysis.

2. *Nomenclature.* The terms solid, surface, plane, sphere, curve, line, etc., will be used with their ordinary significations in Euclidean geometry.

By space, without qualification, is meant Euclidean space of three dimensions. When spoken of as a locus it may be conveniently called a Euclidean.

By the director of a point, or the quaternion of a point, is meant the directed straight line drawn from the origin to the point. Geometrically interpreted, a quaternion is a director.

The word perpendicular, when not qualified, is used in its ordinary sense; thus two planes are perpendicular to one another when a straight line can be found in one of them which is perpendicular to every straight line in the other.

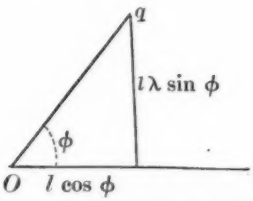
Two planes are said to be hyperperpendicular to one another when every straight line in the one plane is perpendicular to every straight line in the other.

In general, and unless specification to the contrary be made, the letters $a, b, c, d, e, p, q, r, s, t, u, v$ denote quaternions, $h, l, m, n, x, y, z, \theta, \phi, \psi, \chi$ scalars, $\alpha, \beta, \gamma, \delta, \rho, \sigma, \tau$ vectors. The four fundamental quaternion units are denoted by $1, i, j, k$ and their geometrical meaning is: four mutually perpendicular directors of unit length. The Hamiltonian notation is employed throughout.

3. *Amplitude.* When the quaternion is written in the form

$$q = l(\cos\phi + \lambda \sin\phi),$$

in which l is its tensor and λ is a unit vector, ϕ is called its amplitude. The scalar part, $l \cos \phi$, regarded as a director, lies along the scalar axis, and the vector part, $l\lambda \sin \phi$, is the director-perpendicular dropped from the extremity of q to the scalar axis; for this axis is, by definition, perpendicular to i, j, and k, and consequently also perpendicular to every vector. The amplitude is



therefore the angle (more strictly the arc-ratio of angle) between the quaternion, regarded as a director, and the scalar axis. When $\phi = \pi/2$, the quaternion director becomes a vector and is at right angles to the scalar axis; when $\phi = 0$, it is a part of the scalar axis itself.

4. *Geometric Addition.* As applied to a series of quaternions, interpreted as directors in a four-dimensional space, the law of geometric addition is: The sum of any two or more quaternion-directors (whether they be vectors, scalars, or any combinations of these) is the director that extends from the initial to the terminal extremity of the zig-zag formed by so disposing the several director-terms of the sum that all their intermediate extremities are conterminous.

By assuming the existence of a fourth independent direction in space, the geometrical interpretation of a quaternion (as given in § 2) makes this law of geometric addition a mere corollary of the law of vector addition in a three-dimensional domain. We require only to reiterate, for quadrinomials of the form $w + ix + jy + kz$, the statements that are valid for trinomials of the form $ix + jy + kz$.

Since quaternions obey the commutative law of addition, several geometrical steps lead to the same position in whatever order they may be taken, and two directors are identical when either one of them can be derived from the other by a simple translation unaccompanied by any rotation. If a rotation is required in order to bring the two directors into coincidence, they are distinguishable from one another by the fact that their versor parts are distinct. This commutative law in geometric addition implies the validity of the so-called parallel axiom and demands the existence of parallel elements (lines, planes, etc.), parallelism being determined by the usual Euclidean criteria.

5. *Relative Direction.* Let the quaternions q, q', which, for convenience and without loss of generality, may have unit tensors, be written in the binomial forms:

$$q = \cos \phi + \lambda \sin \phi,$$
$$q' = \cos \phi' + \lambda' \sin \phi',$$

and denote by θ the angle they form with one another. The ratio of q to q' is

$$q/q' = (\cos\phi + \lambda \sin\phi)(\cos\phi' - \lambda' \sin\phi'),$$

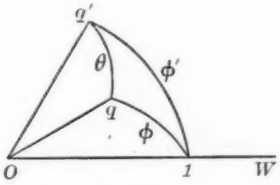
or, in its developed form,

$$q/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi' S\lambda\lambda' + \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi' V\lambda\lambda',$$

and

$$Sq/q' = \cos\phi\cos\phi' - \sin\phi\sin\phi' S\lambda\lambda' = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

We may represent the arcs ϕ, ϕ', θ by the sides of a spherical triangle on the surface of a unit sphere. In such a figure the intersection of the planes of the



arcs ϕ, ϕ' is the scalar axis and the angle of the vectors λ, λ' is the angle between these two planes. Hence, by the cosine formula of spherical trigonometry,

$$\cos\theta = \cos\phi\cos\phi' + \sin\phi\sin\phi'\cos(\lambda, \lambda').$$

Thus the measure of the inclination of q to q' is

$$Sq/q' = \cos\theta.*$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that $\theta = \pi/2$ or an odd multiple of $\pi/2$. Hence:

The necessary and sufficient condition in order that two directors q, q' may be perpendicular to one another is that their ratio shall be a vector. Symbolically,

$$S \cdot q K q' = 0.$$

To obtain the condition of parallelism we note that the vector part of q/q' is

$$Vq/q' = \lambda\sin\phi\cos\phi' - \lambda'\sin\phi'\cos\phi - \sin\phi\sin\phi' V\lambda\lambda',$$

and that q and q' are parallel (or identical) if, and only if, their versor parts are either identical, or differ only in algebraic sign; that is, if

$$\phi' = \phi + n\pi, \quad \lambda' = \lambda, \quad V\lambda\lambda' = 0,$$

where $n = 0$, or a positive integer. But these conditions are fully expressed in the equation

* Cf. HATHAWAY: loc. cit. (1) p. 55.

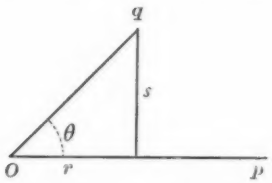
$$\mathrm{V}q/q' = \pm\lambda \sin(\phi - \phi') = 0\,.$$

Hence, the necessary and sufficient condition in order that two directors q, q' may be parallel is that their ratio shall be a scalar. Symbolically,

$$\mathrm{V}\cdot q\mathrm{K}q' = 0\,,$$

In the conditions $\mathrm{S}\cdot q\mathrm{K}\rho = 0$, $\mathrm{V}\cdot q\mathrm{K}\rho = 0$, the requirement that ρ and q shall have unit tensors is obviously not necessary.

6. *Binomial Form of q/ρ.* Let θ be the angle between any two quaternion directors ρ, q, let s be the director-perpendicular dropped from the terminal ex-



tremity of q to ρ, and r the director from the origin (intersection of ρ, q) to the foot of this perpendicular. Then, by geometric addition,

$$q = r + s$$

and

$$q/\rho = r/\rho + s/\rho\,;$$

and s/ρ, being the ratio of a pair of mutually perpendicular directors, is a vector. Hence, if ρ be defined as the ratio $\mathrm{U}s/\mathrm{U}\rho$, it is a unit vector and

$$q/\rho = \frac{\mathrm{T}q}{\mathrm{T}\rho}(\cos\theta + \rho\sin\theta)\,.$$

7. *Projection.* It is now evident that

$$\mathrm{S}\cdot q\mathrm{K}\rho = \mathrm{T}q\mathrm{T}\rho\cos(\rho, q)\,,$$

and that

$$\mathrm{TV}\cdot q\mathrm{K}\rho = \mathrm{T}q\mathrm{T}\rho\sin(\rho, q)\,;$$

and since obviously

$$\mathrm{S}(\mathrm{V}\cdot q\mathrm{K}\rho)(\mathrm{U}\rho\cdot\mathrm{K}\rho) = 0\,,$$

$(\mathrm{V}\cdot q\mathrm{K}\rho)\mathrm{U}\rho$ is perpendicular to ρ. Hence the identity

$$q \equiv (\mathrm{S}\cdot q\mathrm{K}\rho)(\mathrm{K}\rho)^{-1} + (\mathrm{V}\cdot q\mathrm{K}\rho)(\mathrm{K}\rho)^{-1}$$

has the following evident interpretation:

$$\frac{\mathrm{S}\cdot q\mathrm{K}\rho}{\mathrm{T}\rho}\mathrm{U}\rho = \text{projection of } q \text{ upon } \rho\,,$$

$$\frac{\mathrm{V}\cdot q\mathrm{K}\rho}{\mathrm{T}\rho}\mathrm{U}\rho = \text{projection of } q \text{ upon a director perpendicular to } \rho\,,$$

and the sum of these two projections is q itself.

8. *Rectangular Coördinates.* The propositions of § 5 might have been proved in terms of rectangular coördinates. Suppose the two quaternions given in the quadrinomial form:

$$q = w + ix + jy + kz,$$
$$q' = w' + ix' + jy' + kz'.$$

Their direction cosines are

$$w/l\ ,\ x/l\ ,\ y/l\ ,\ z/l\ ,$$
$$w'/l',\ x'/l',\ y'/l',\ z'/l',$$

where

$$l = \sqrt{w^2 + x^2 + y^2 + z^2},$$
$$l' = \sqrt{w'^2 + x'^2 + y'^2 + z'^2},$$

and the formula for the cosine of the angle between them is

$$\cos\theta = \frac{ww' + xx' + yy' + zz'}{ll'}.$$

The ratio of q to q' may be written in the form

$$q/q' = \frac{(w + ix + jy + kz)(w' - ix' - jy' - kz')}{w'^2 + x'^2 + y'^2 + z'^2}.$$

In the numerator of this fraction the coefficients of 1, i, j, and k are respectively,

$$W = ww' + xx' + yy' + zz',$$
$$X = w'x - wx' + yz' - y'z,$$
$$Y = w'y - wy' + zx' - z'x,$$
$$Z = w'z - wz' + xy' - x'y;$$

and in terms of these q/q' takes the form

$$q/q' = \frac{W + iX + jY + kZ}{l'^2}.$$

In order that q and q' may be perpendicular to one another it is necessary and sufficient that

$$ww' + xx' + yy' + zz' = 0, \quad \text{i. e.,} \quad W = 0,$$

or otherwise expressed,

$$q/q' = \text{a vector.}$$

In order that q and q' may be parallel to one another it is necessary and sufficient that

$$\frac{w}{w'} = \frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'},$$

that is,

$$iX + jY + kZ = 0,$$

or otherwise expressed,

$$q/q' = \text{a scalar.}$$

9. *Equations of Loci.* In general it will be found that a quaternion equation in one variable has a definite, finite number of solutions and that, if it represent a locus, its coefficients must satisfy certain conditions, a remark in fact that is justified by the observation that the quaternion equation

$$f(q) = 0$$

is equivalent to a system of four scalar equations in four scalar variables. Thus a principal first step in the interpretation of the linear equation as a locus will be the assigning of the conditions necessary and sufficient in order that the equation may have an infinite number of solutions.

10. *Transference of Origin.* By virtue of the law of geometric addition, a locus, given in terms of a variable quaternion p, may be referred to a new origin, whose director is c, and be represented by an equation in terms of a new variable quaternion q, by substituting

$$p = c + q$$

in the given equation. Thus the equation

$$ap + pa^{-1}aa + 2a = 0 \qquad (a^2 = -1)$$

is satisfied by the value $p = aa$ and is referred to a point in the locus itself, as a new origin, by writing

$$p = aa + q;$$

and through this substitution it becomes

$$aq + qa^{-1}aa = 0.$$

In comparing two or more loci, advantage may be taken of this principle in order to simplify one or more of the equations by transferring the origin to a point within one of the loci, or to a point common to two or more of them.

11. *Solution of* $a_1 p + p a_2 = c$. I indicate briefly HAMILTON'S solution of this equation.* Denote the conjugate of a_2 by $\bar{a}_2$† and multiply $a_1 p + p a_2$ progressively by a_1, regressively by $\bar{a}_2$ and add; there results

* *Lectures on Quaternions*, p. 565, or TAIT'S *Treatise on Quaternions*, 3d ed., p. 136.

† This notation is used throughout the paper.

$$\{a_1^2 + a_1(a_2 + \bar{a}_2) + a_2\bar{a}_2\}\, p = a_1 c + c\bar{a}_2,$$

or

$$p = (a_1^2 + 2a_1 \mathrm{S}a_2 + a_2\bar{a}_1)^{-1}(a_1 c + c\bar{a}_2).$$

If nullitats (quaternions involving scalar $\sqrt{-1}$ and having zero tensors) be excluded the equation has in general but one root. For, denote any root by b; then

$$a_1 b + b a_2 = c,$$

and the original equation may be written in the form

$$a_1(p - b) + (p - b)a_2 = 0;$$

and if p be supposed to have any other value than b, for such value $p - b$ does not vanish and

$$a_1 = -(p - b)a_2(p - b)^{-1}$$

is not indeterminate; therefore

$$\mathrm{S}a_1 = -\mathrm{S}a_2, \quad \mathrm{T}a_1 = \mathrm{T}a_2.$$

Thus the coefficients are not independent and the equation may be written

$$a_1 p + p a_2 = c,$$

in which $a_1 = \mathrm{V}a_1$, $a_2 = \mathrm{V}a_2$, and $\mathrm{T}a_1 = \mathrm{T}a_2$.

It follows that unless the conditions: $\mathrm{S}a_1 = -\mathrm{S}a_2$, $\mathrm{T}a_1 = \mathrm{T}a_2$, be satisfied, the equation has but one root. But if it have more than one, it then has an infinite number of roots and represents a locus.

12. As an example of other equations in a variable quaternion p whose solution leads to this linear bilateral form, consider

$$\mathrm{S}ap = m, \quad \mathrm{S}bp = n,$$

which obviously represent a pair of Euclideans (ordinary spaces of three dimensions). They are equivalent to

$$ap + \bar{p}\bar{a} = 2m, \quad bp + \bar{p}\bar{b} = 2n,$$

from which is obtained, by a series of sufficiently simple operations,

$$\bar{b}ap - pb\bar{a} = 2(m\bar{b} - n\bar{a}),$$

and since $\mathrm{S}\bar{b}a = \mathrm{S}b\bar{a}$ and $\mathrm{T}\bar{b}a = \mathrm{T}b\bar{a} = \mathrm{T}ab$, this may be written

$$\mathrm{UV}\bar{b}a \cdot p - p \cdot \mathrm{UV}b\bar{a} + \frac{2(n\bar{a} - m\bar{b})}{\mathrm{TV}ab} = 0.$$

Here again is a single equation in a variable quaternion p representing a locus, viz., the intersection of a pair of Euclideans.

§§ 13–16. Loci of Linear Equations.

13. *Planes through the Origin.* If a_1 and a_2 be any two unit vectors and p a variable quaternion such that

$$a_1 p + p a_2 = 0,$$

then

$$\mathrm{S}(a_1 + a_2)\mathrm{V}p = 0.$$

Thus, under the conditions imposed, $\mathrm{V}p$ moves in a plane through the origin perpendicular to $a_1 + a_2$ and can therefore be written in the form

$$\mathrm{V}p = x(a_1 - a_2) + y\mathrm{V}a_1 a_2,$$

in which x and y are independent variable scalars; thus p has the form

$$p = x(a_1 - a_2) + y\mathrm{V}a_1 a_2 + \mathrm{S}p.$$

Subject this expression for p to the condition that it satisfy the equation $a_1 p + p a_2 = 0$ and note that the part involving $x(a_1 - a_2)$ vanishes identically. There remains

$$a_1(y\mathrm{V}a_1 a_2 + \mathrm{S}p) + (y\mathrm{V}a_1 a_2 + \mathrm{S}p)\, a_2 = 0,$$

and this solved for $\mathrm{S}p$ gives

$$\mathrm{S}p = y(a_1 - a_2)(a_1 + a_2)^{-1}\mathrm{V}a_1 a_2 = y(1 + \mathrm{S}a_1 a_2),$$

whence

$$p = x(a_1 - a_2) - y a_1(a_1 - a_2).$$

Thus, as x and y vary p moves in the plane of the two directors $a_1 - a_2$, $a_1(a_1 - a_2)$ and the equation of this plane is

$$a_1 p + p a_2 = 0.$$

Or this equation may be written in the form

$$a_1 p - p a^{-1} a_1 a = 0,$$

where a is any quaternion satisfying the condition $a_1 a + a a_2 = 0$.

14. *Planes through any Point.* If q satisfy the equation

$$a_1 q + q a_2 + 2a = 0, \qquad (a_1^2 = a_2^2 = -1),$$

it is immediately evident that $a_1 a = a a_2$ and that therefore

$$a_1 \cdot a_1 a + a_1 a \cdot a_2 + 2a = 0.$$

Hence the values of q that satisfy this equation can be written in the form

$$q = a_1 a + x(a_1 - a_2) - y a_1(a_1 - a_2), \tag{1}$$

where x and y are arbitrary scalars. The equation represents a plane through the extremity of $a_1 a$.

* Stringham: loc. cit. (1) (1884), pp. 54–55.

These values of q may also be expressed in either of the forms:

$$q = a_1 a + l a_1^z (a_1 - a_2),$$
$$q = a_1 a + a_1 r - r a_2,$$

where l and z are arbitrary scalars and r is an arbitrary quaternion. The verifications are easily made by substitution in the equation.

It is important to observe that since $a_1 a = a a_2$, only two constants, a unit vector and a quaternion, are necessary for the complete determination of a plane. Thus, these constants being a_1 and a, the equation is

$$a_1 \cdot p + p \cdot a^{-1} a_1 a + 2a = 0.$$

15. *Planes containing given Elements.* Such an equation as the one last written represents any plane in four-dimensional space. For, suppose the plane to be determined by a point a_0 and two straight lines c, e through this point. We may find two unit vectors a_1, a_2 such that

$$a_1 c + c a_2 = 0, \quad a_1 e + e a_2 = 0.$$

In fact the solution of these equations for a_1, a_2 is

$$a_1 = \pm \mathrm{UV} e\bar{c}, \quad a_2 = \pm \mathrm{UV} \bar{e} c,$$

and by substitution in the equations it is found that these versors must have like signs. The vectors a_1, a_2 being thus determined, let $a = -a_1 a_0$ (equivalent to $a_0 = a_1 a$) and consider the equation

$$a_1 p + p a_2 + 2a = 0.$$

It is the equation of a plane and it is satisfied by a_0, by $a_0 + c$ and by $a_0 + e$, and thus, as required, it represents the plane that contains the point a_0 and the lines c, e.

If the plane be determined by three points, c, d, e, the equations of condition for a_1, a_2 may be written in the form

$$a_1(c - d) + (c - d) a_2 = 0, \quad a_1(c - e) + (c - e) a_2 = 0,$$

and then

$$a_1 = \mathrm{UV}(cd + d\bar{e} + \bar{e}c), \quad a_2 = \mathrm{UV}(\bar{c}d + d\bar{e} + e\bar{c}),$$
$$a = -\tfrac{1}{2}(a_1 c + c a_2) = -\tfrac{1}{2}(a_1 d + d a_2) = -\tfrac{1}{2}(a_1 e + e a_2).$$

16. *The director $a_1 a$ is perpendicular to the plane.* For since $a_1 a = a a_2$,

$$\mathrm{S} a_1 a \{x(a_2 - a_1) - y(a_1 - a_2) a_1\} = 0,$$

the condition for perpendicularity (§ 5). Thus, the equation of a plane through the extremity of, and perpendicular to, a given director a_0 is

$$a_1 q + q a_2 - 2 a_1 a_0 = 0,$$

in which a_1, a_2 are any two unit vectors, satisfying the condition $a_1 a_0 = a_0 a_2$.

It will be convenient to write $(a_1, a_2, 2a)$ as an abbreviation meaning: the plane whose equation is $a_1 q + q a_2 + 2a = 0$ $(a_2 = a^{-1} a_1 a)$.

§§ 17–22. Intersections.

17. *Director of Meeting-point.* It is here shown that two planes,

$$a_1 p + p a_2 + 2a = 0, \quad \beta_1 p + p \beta_2 + 2b = 0,$$

always meet in at least one point, and the director to this point is found in terms of the constants of the equation. Operate progressively and regressively on the first equation with β_1, β_2, on the second with a_1, a_2 and in each case take the difference of the products and then add these differences together; the result is

$$2p\mathrm{S}(a_1\beta_1 - a_2\beta_2) + 2(\beta_1 a - a\beta_2 + a_1 b - b a_2) = 0.$$

Hence, in general

$$p = \frac{\beta_1 a - a\beta_2 + a_1 b - b a_2}{\mathrm{S}(a_2\beta_2 - a_1\beta_1)}.$$

If the planes have two points in common they have a line of intersection and a relation between the coefficients is obviously necessary. Hence, in general, two planes meet in one and only one point.

18. *Intersection Lines at the Origin.—In order that the two planes* $(a_1, a_2, 0)$, $(\beta_1, \beta_2, 0)$, *which meet at the origin, may meet in a straight line, it is necessary and sufficient that* $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$.

It is necessary; for, if there be some value of p not zero that satisfies the two equations $a_1 p + p a_2 = 0$, $\beta_1 p + p\beta_2 = 0$, then

$$a_1 = -p a_2 p^{-1}, \quad \beta_1 = -p\beta_2 p^{-1},$$

and therefore

$$\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2.$$

It is sufficient; for, if

$$c = \beta_1(x - y a_1)(a_1 + a_2) - (x - y a_1)(a_1 + a_2)\beta_2,$$

where x and y are arbitrary scalars, then the condition $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$ suffices to make $a_1 c + c a_2 = 0$, $\beta_1 c + c\beta_2 = 0$, and thus the two planes meet in c.

19. *Cosine of a Dihedral Angle.* Two planes are supposed to meet in a straight line at the origin. Let u, v be two unit directors drawn, u in the plane $(a_1, a_2, 0)$, v in the plane $(\beta_1, \beta_2, 0)$ perpendicular to their line of intersection. Then $a_1 u$ lies in $(a_1, a_2, 0)$ and $\beta_1 v$ in $(\beta_1, \beta_2, 0)$, and identically

$$\mathrm{S}a_1 u\bar{u} = \mathrm{S}\beta_1 v\bar{v} = 0,$$

conditions sufficient to make $a_1 u$ and $\beta_1 v$ perpendicular respectively to u and v.

Hence a_1u and β_1v are two expressions for the intersection line of the two planes and $a_1u = \pm\beta_1v$, whence

$$\mathrm{S}u\bar{v} = \pm\,\mathrm{S}a_1\beta_1 = \pm\,\mathrm{S}a_2\beta_2\,. \qquad (\S\,18)$$

These are the cosines of the dihedral angles formed by the two planes.

The planes are perpendicular to one another if $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2 = 0$.

20. *Intersection Lines in General.—In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may meet in a straight line it is necessary and sufficient that*

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0\,.$$

If the two planes meet in a straight line at a finite distance it will be possible to transfer the origin to a point in this line and thus reduce the equations to the form

$$a_1q + qa_2 = 0\,, \quad \beta_1q + q\beta_2 = 0\,.$$

Hence (§18) a necessary condition for a straight-line intersection is $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$. But when this condition is satisfied the expression (§ 17)

$$\frac{a_1b - ba_2 + \beta_1a - a\beta_2}{\mathrm{S}(a_2\beta_2 - a_1\beta_1)}\,,$$

for a *point* of intersection of the two planes, is determinate and infinite unless the numerator vanish. Hence the condition

$$a_1b - ba_2 + \beta_1a - a\beta_2 = 0$$

is necessary.

This condition requires that $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$, but the converse of this statement is not true; for if

$$f = a_1b - ba_2\,, \quad g = \beta_1a - a\beta_2\,, \quad m = \mathrm{S}(a_2\beta_2 - a_1\beta_1)\,,$$

then the identical relations

$$a_1f + fa_2 = 0\,, \quad a_1g + ga_2 + 2am = 0\,,$$

$$\beta_1g + g\beta_2 = 0\,, \quad \beta_1f + f\beta_2 + 2bm = 0\,,$$

put in immediate evidence the fact that, x being any scalar, $f = xg$ makes $m = 0$, and *vice versa*, but $m = 0$ does not involve $f + g = 0$.

The sufficiency of the condition $f + g = 0$ will now be proved if it be shown that the vanishing of $f + g$ makes it possible to assign a singly infinite series of quaternion directors which satisfy the equations of the two planes. But identically,

$$a_1\mathrm{V}a\bar{b}\cdot f + \mathrm{V}a\bar{b}\cdot fa_2 = af\bar{f}\,,$$

$$\beta_1\mathrm{V}b\bar{a}\cdot g + \mathrm{V}b\bar{a}\cdot g\beta_2 = bg\bar{g}\,,$$

and, if $f + g = 0$, also

$$\beta_1\mathrm{V}a\bar{b}\cdot f + \mathrm{V}a\bar{b}\cdot f\beta_2 = bf\bar{f}\,,$$

$$a_1xf + xfa_2 = 0\,, \quad \beta_1xf + xf\beta_2 = 0\,,$$

where x is an arbitrary scalar. Hence, *provided f and g be not separately zero*, $(x - 2\mathrm{V}ab)/f$ represents a singly infinite series of quaternions possessing the required property.

Thus the condition $f = -g \neq 0$ determines that the two planes meet in a straight line and enables us to assign as its equation

$$\rho = (x - 2\mathrm{V}ab)/f.$$

Here neither f nor $\mathrm{V}ab$ can vanish separately; for if $b = ya$, then $f = y(a_1a - aa_2) = 0$, and if $f = 0$ both a and b represent the same director, namely the intersection of the two planes $(a_1, -a_2, 0)$, $(\beta_1, -\beta_2, 0)$, which can only happen when b is a numerical multiple of a.

Suppose then that $f = g = 0$; there are two cases:

(1) $a_1b = ba_2 \neq \beta_1a = a\beta_2$; (2) $a_1b = ba_2 = \beta_1a = a\beta_2$.

(1) If $a_1b = ba_2 \neq \beta_1a = a\beta_2$, then the above expression for ρ may be replaced by

$$\rho = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1a - a_1b + xa)$$

which, as may be easily verified, reduces the equations of $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to identities for all (scalar) values of x. This is again the equation of a straight line. (I have not proved that $\mathrm{V}a_1\beta_1$ is not zero.)

(2) If $a_1b = ba_2 = \beta_1a = a\beta_2$, a formal solution is

$$\rho = (\mathrm{V}a_1\beta_1)^{-1}(\beta_1 + x)a,$$

an expression which again reduces the equations of the two planes to identities. But here, since b is a multiple of a and since now $a_1\beta_1 = -ab^{-1}$, we have $\mathrm{V}a_1\beta_1 = 0$ and therefore

$$\beta_1 = \pm a_1, \quad \beta_2 = \pm a_2,$$

and the signs must be $++$ or $--$. Hence $\rho = \infty$. This is the case of parallel planes, as will be shown in the sequel (§ 29).

Thus, in whatever way $f + g$ becomes zero the two planes meet in a straight line, either at a finite or at an infinite distance.

Incidentally, either $-2\mathrm{V}ab/f$ or $(\mathrm{V}a_1\beta_1)^{-1}(\beta_1a - a_1b)$, the former in the general case, the latter when $a_1b = ba_2 \neq \beta_1a = a\beta_2$, is determined as the director-perpendicular from the origin to the intersection-line of the two planes. This is made evident by applying the test of (§ 5).

21. *Intersections at Infinity.* If the two planes meet only once at an infinite distance, we must have $(f + g)/m = 0$, and $f + g \neq 0$ (§ 20), and these conditions are obviously sufficient. Hence, taking account of § 20 (2):

For the two planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ to meet at infinity, the necessary and sufficient conditions are:

if at a point, $\mathrm{S}a_1\beta_1 = \mathrm{S}a_2\beta_2$, $f + g \neq 0$;

if in a line, $\beta_1 = \pm a_1$, $\beta_2 = \pm a_2$.

22. *Normals.* It is easily verified that:

(1) The perpendicular distance (director-normal) between the two planes $(a_1, a_2, 2a)$ and $(a_1, a_2, 2b)$ is, in both magnitude and direction, $a_1(a - b)$.

(2) The director normal from the extremity of c to the plane $(a_1, a_2, 2b)$ is

$$\tfrac{1}{2}a_1(2a + a_1c + ca_2).$$

(3) It has been shown in § 21 that the director-normal from the origin to the intersection of $(a_1, a_2, 2a)$ and $(\beta_1, \beta_2, 2b)$ is

$$(ab - ba)(ba_1 - a_2b)$$

the conditions $f + g = 0$, $f \neq g$ being here essential.

§§ 23–27. Divergence of Two Planes and its Measure.

23. *Isoclinal Angles.* Two planes meet, in general, in a point or in a straight line (§§ 17, 20). It is always possible, as will presently appear, to pass through any point common to two given planes other (transversal) planes which meet the former in straight lines and form with them equal opposite interior dihedral angles. The plane angle formed by the edges of the two dihedrals, whose plane as viewed from a point between the two given planes is equally inclined to them, will be called their *isoclinal angle.* This angle is an appropriate index (measure) of the amount (rapidity) of the divergence of the two planes under a special aspect, namely, from the vertex and in the direction of the sides of the isoclinal angle. Its variations afford the means for determining certain important relations of planes to one another.

The given planes being $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$, we transfer the origin to the point of, or to a point in, their intersection. Their equations become (§ 13)

$$a_1p + pa_2 = 0, \quad \beta_1p + p\beta_2 = 0;$$

and x, y being continuous scalar variables,

$$u = a_1^x(a_1 - a_2), \quad v = \beta_1^y(\beta_1 - \beta_2)$$

represent two directors lying in them, u in the first, v in the second (§ 14). I enquire whether and under what conditions the plane angle formed by u and v may have a maximal or a minimal value. Without loss of generality we may suppose that $\mathrm{T}u = \mathrm{T}v = 1$.

The first and second partial derivatives of $\mathrm{S}u\bar{v}$ with respect to x and y are

$$\frac{\partial \mathrm{S}u\bar{v}}{\partial x} = \frac{\pi}{2}\mathrm{S}a_1u\bar{v}, \quad \frac{\partial \mathrm{S}u\bar{v}}{\partial y} = -\frac{\pi}{2}\mathrm{S}\beta_1u\bar{v},$$

$$\frac{\partial^2 \mathrm{S}u\bar{v}}{\partial x^2} = -\frac{\pi^2}{4}\mathrm{S}u\bar{v}, \quad \frac{\partial^2 \mathrm{S}u\bar{v}}{\partial y^2} = -\frac{\pi^2}{4}\mathrm{S}u\bar{v},$$

and

$$\frac{\partial^2 \mathrm{S}u\bar{v}}{\partial x\,\partial y} = -\frac{\pi^2}{4}\mathrm{S}\beta_1a_1u\bar{v}.$$

The two second partial derivatives, with respect to x, and with respect to y, being negative as compared with the function itself, the conditions for a maximum or a minimum of $S u\bar{v}$ are

$$Sa_1 u\bar{v} = 0, \quad S\beta_1 u\bar{v} = 0,$$

and

$$S^2 u\bar{v} > S^2\beta_1 a_1 u\bar{v}, \quad \text{or} \quad S^2 u\bar{v} < S^2\beta_1 a_1 u\bar{v}.$$

Thus $Su\bar{v}$, *and therefore also the angle of* u, v, *will have maxima and minima if there exist an inequality of the form:*

$$|Su\bar{v}| \neq |Sa_1 u\beta_1 \bar{v}|.$$

Let it be supposed, for the moment, that the directors u, v are fixed by the conditions thus imposed (it will be shown presently that this is actually the case) and let u and v be turned in their respective planes about the origin through the same angle $\frac{1}{2}\pi\theta$ by multiplying the first by a_1^θ, and the second by β_1^θ, where θ is a scalar variable.

(1) The essential preliminary condition: *The plane of* $a_1^\theta u$ *and* $\beta_1^\theta v$ *shall intersect both* $(a_1, a_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *in straight lines*, is here fulfilled; for γ_1, γ_2 being $= UVa_1^\theta u\bar{v}\beta_1^{-\theta}$, $UV\bar{u}a_1^{-\theta}\beta_1^\theta v$, the equation of this plane is, by § 15 (1),

$$\gamma_1 p + p\gamma_2 = 0,$$

and the sufficient conditions (§ 18)

$$Sa_1\gamma_1 = Sa_2\gamma_2, \quad S\beta_1\gamma_1 = S\beta_2\gamma_2,$$

are here satisfied.

(2) *The angle of* $a_1^\theta u$ *and* $\beta_1^\theta v$ *is isoclinal to* $(a_1, a_2, 0)$ *and* $(\beta_1, \beta_2, 0)$ *for all values of* θ. For we may write

$$a_1^\theta = x + ya_1, \quad \beta_1^\theta = x + y\beta_1 \qquad (x^2 + y^2 = 1),$$

and then, the conditions $Sa_1 u\bar{v} = S\beta_1 u\bar{v} = 0$ being satisfied,

$$Sa_1\gamma_1 = xyS(\beta_1 a_1 u\bar{v} - u\bar{v}) = -S\beta_1\gamma_1$$
$$= Sa_2\gamma_2 = -S\beta_2\gamma_2,$$

that is, the corresponding exterior-interior dihedral angles on the same side of the transversal are equal. This isoclinal angle may thus be treated as a function of θ.

(3) Let the values of θ that make the angle of $a_1^\theta u$ and $\beta_1^\theta v$ a maximum or a minimum be sought. The first and second derivatives of $Sa_1^\theta u\bar{v}\beta_1^{-\theta}$ with respect to θ are

$$\frac{dSw}{d\theta} = \frac{\pi}{2} S(a_1 - \beta_1)w \qquad (w = a_1^\theta u\bar{v}\beta_1^{-\theta}),$$

$$\frac{d^2 Sw}{d\theta^2} = -\frac{\pi^2}{2} S(1 + \beta_1 a_1)w,$$

and by virtue of the previously assumed conditions $Sa_1u\bar{v} = S\beta_1u\bar{v} = 0$, the first derivative vanishes for all integral values of θ.

When $\theta = 0$ or an even number, the second derivative is a negative quantity if $Su\bar{v} > Sa_1u\overline{\beta_1v}$, and these (as previously found) are the conditions that make $Su\bar{v}$ a maximum.

When $\theta =$ an odd number, the second derivative is a positive quantity if $Su\bar{v} > Sa_1u\overline{\beta_1v}$, and these are the conditions that make $Su\bar{v}$ a minimum.

It is immediately evident that if there be maxima and minima, there are two of each, occurring alternately at intervals of ninety degrees; for the successive angles are:

$$\angle(a_1^\theta u,\ \beta_1^\theta v) \qquad (\theta = 0,\ 1,\ 2,\ 3),$$

and $Sa_1^\theta u\bar{u} = S\beta_1^\theta v\bar{v} = 0$ if θ be an odd integer.

It will be shown presently (§ 24) that non-integral values of θ do not give rise to maximal or minimal isoclinal angles. The above enumeration is therefore exhaustive.

(4) *The two planes of the maximal and minimal isoclinal angles, and these only, are orthogonal to both of the given planes.* For, in order that the conditions for perpendicularity (§ 19) may be satisfied, namely:

$$Sa_1\gamma_1 = Sa_2\gamma_2 = S\beta_1\gamma_1 = S\beta_2\gamma_2 = 0,$$

it is necessary and sufficient that x or $y = 0$ (in $a_1^\theta = x + ya_1$, $\beta_1^\theta = x + y\beta_1$, of (2)), and these are precisely the condition $\theta =$ an integer.

(5) If the terms of $Su\bar{v} > Sa_1u\overline{\beta_1v}$ be interchanged, $Su\bar{v}$ becomes a minimum and $Sa_1u\overline{\beta_1v}$ a maximum. Such an interchange, however, is merely equivalent to assigning $u' = a_1u$, $v' = \beta_1v$ as the two director boundaries of the minimal and a_1u', β_1v' as the director boundaries of the maximal angle. And unless $Su\bar{v} = Sa_1u\overline{\beta_1v}$, an inequality of the form $Su\bar{v} > Sa_1u\overline{\beta_1v}$ may be assumed to exist. Hence if the conditions

$$Sa_1u\bar{v} = 0,\quad S\beta_1u\bar{v} = 0,\quad Su\bar{v} \neq Sa_1u\overline{\beta_1v}$$

be satisfied, either $Su\bar{v}$ is a maximum and $Sa_1u\overline{\beta_1v}$ a minimum, or *vice versa*.

(6) It remains to show that the conditions $Sa_1u\bar{v} = S\beta_1u\bar{v} = 0$ suffice to determine a convenient measure of the divergence of the two planes. In another form, $Sa_1u\bar{v} = 0$ is

$$a_1u\bar{v} - v\bar{u}a_1 = 0,$$

which is also

$$\beta_1a_1u\bar{v} - \beta_1v\bar{u}a_1 = 0.$$

But

$$\beta_1a_1u\bar{v} + v\bar{u}a_1\beta_1 = 2S\beta_1a_1u\bar{v} = \beta_1v\bar{u}a_1 + v\bar{u}a_1\beta_1.$$

Multiplied by $u\bar{v}$ this last equation is

$$2u\bar{v}\mathrm{S}\beta_1 a_1 u\bar{v} = u\bar{v}v\beta_2 a_2\bar{u} + v\bar{v}u\bar{u}a_1\beta_1,$$

whence, by equating scalar parts and dividing by 2,

$$\mathrm{S}u\bar{v}\cdot\mathrm{S}a_1 u\beta_1 v = -\tfrac{1}{2}\mathrm{S}(a_1\beta_1 + a_2\beta_2).^*$$

There is thus obtained, as the measure of divergence of the two planes, the product of the cosines of their numerical maximal and minimal isoclinal angles.

24. *The Ultimate Criteria of Maxima and Minima.* The higher odd-order derivatives of $\mathrm{S}w$, $= \mathrm{S}a_1^\theta u\bar{v}\beta_1^{-\theta}$ are

$$\frac{d^{2n-1}\mathrm{S}w}{d\theta^{2n-1}} = (-1)^{n-1}\frac{\pi^{2n-1}}{2}\mathrm{S}(a_1 - \beta_1)w,$$

and the even-order derivatives are

$$\frac{d^{2n}\mathrm{S}w}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}\mathrm{S}(1 + \beta_1 a_1)w.$$

If we write

$$(a_1^\theta,\ \beta_1^\theta) = (x + ya_1,\ x + y\beta_1) \qquad (x^2 + y^2 = 1),$$

and maintain intact the conditions $\mathrm{S}a_1 u\bar{v} = \mathrm{S}\beta_1 u\bar{v} = 0$, these higher derivatives become

$$\frac{d^{2n-1}\mathrm{S}w}{d\theta^{2n-1}} = (-1)^{n-1}\pi^{2n-1}xy\mathrm{S}(a_1 u\overline{\beta_1 v} - u\bar{v}),$$

$$\frac{d^{2n}\mathrm{S}w}{d\theta^{2n}} = (-1)^n\frac{\pi^{2n}}{2}(y^2 - x^2)\mathrm{S}(a_1 u\overline{\beta_1 v} - u\bar{v}).$$

Hence the derivatives of odd order vanish if, and only if, $x = 0$, or $y = 0$, or $\mathrm{S}a_1 u\overline{\beta_1 v} = \mathrm{S}u\bar{v}$; and if $\mathrm{S}a_1 u\overline{\beta_1 v} = \mathrm{S}u\bar{v}$ all the derivatives of even order vanish. The immediate consequence is:

Non-integral values of θ (corresponding to $x \neq 0$, $y \neq 0$) give rise to neither maxima nor minima of the isoclinal angle $(a_1^\theta u, \beta_1^\theta v)$ and the enumeration of § 23 (3) *is complete.*

If the equations of the two planes be in the general form, referred to an arbitrary origin, the common vertex of their isoclinal angles is their point of meeting, or a point in their line of intersection, and c being the director to this point, we may assign as general solutions of their equations: $c + a_1^\theta u$ and $c + \beta_1^\theta v$ respectively, where u and v are determined by the foregoing conditions for maxima and minima. The final form of our criterion then is:

The conditions necessary and sufficient in order that the two planes

* STRINGHAM: loc. cit. (3) (1888), p. 64. Of course the formulæ leading up to this result had been determined prior to the date August, 1888. Cf. HATHAWAY: loc. cit. (2) (1897), § 45, p. 10.

$(a_1, \beta_1, 2a)$, $(\beta_1, \beta_2, 2b)$ may have maximal and minimal isoclinal angles are: that there exist solutions $c + a_1^e u$ and $c + \beta_1^e v$, of their respective equations, such that

$$\mathrm{S}a_1 u\bar{v} = 0\,, \quad \mathrm{S}\beta_1 u\bar{v} = 0\,, \quad \mathrm{S}u\bar{v} \neq \mathrm{S}a_1 u\beta_1 v\,.$$

25. *The Maximal and Minimal Angles Determined.* It has been seen (§ 23 (4)) that the planes of the maximal and minimal isoclinal angles, and these only, cut both the given planes orthogonally. But the conditions of perpendicularity will be here satisfied by writing as the equations of these two orthogonal transversals,

$$\gamma_1 p + p\gamma_2 = 0, \quad \gamma_1 p - p\gamma_2 = 0\,,$$

where

$$\gamma_1 = \mathrm{UV}a_1\beta_1, \quad \gamma_2 = \mathrm{UV}a_2\beta_2\,.$$

Hence the four intersections of these two planes with $(a_1, a_2, 0)$ and $(\beta_1, \beta_2, 0)$ must be the directors hitherto designated by u, v, $a_1 u$, $\beta_1 v$; and we may write (§ 18)

$$u = a_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)a_2\,,$$

$$v = \beta_1(\gamma_1 + \gamma_2) - (\gamma_1 + \gamma_2)\beta_2\,,$$

$$a_1 u = u' = a_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)a_2\,,$$

$$\beta_1 v = v' = \beta_1(\gamma_1 - \gamma_2) - (\gamma_1 - \gamma_2)\beta_2\,;$$

and we may verify independently the necessary conditions

$$\mathrm{S}a_1 u\bar{v} = 0\,, \quad \mathrm{S}\beta_1 u\bar{v} = 0\,, \quad \mathrm{S}a_1 u'\bar{v}' = 0\,, \quad \mathrm{S}\beta_1 u'\bar{v}' = 0\,,$$

which are in fact identities if u, v, u', v' have the values above assigned.

The further conditions that u, v shall be perpendicular to u', v' (§ 23 (3)) are here also satisfied, for the planes $(\gamma_1, \gamma_2, 0)$, $(\gamma_1, -\gamma_2, 0)$ are hyperperpendicular to one another (§ 28); or it may be verified independently that

$$\mathrm{S}u\bar{u}' = 0\,, \quad \mathrm{S}v\bar{v}' = 0\,.$$

This determination fails if either $\beta_1 = \pm a_1$ or $\beta_2 = \pm a_2$, for then either $\gamma_1 = 0$, or $\gamma_2 = 0$; but these are the conditions for the failure of maxima and minima (§ 26).

26. *Failure of the Condition* $\mathrm{S}u\bar{v} \neq \mathrm{S}a_1 u\beta_1 v$. From the conditions:

$$\mathrm{S}a_1 u\bar{v} = \mathrm{S}a_2\bar{v}u = 0\,, \quad \mathrm{S}\beta_1 u\bar{v} = \mathrm{S}\beta_2\bar{v}u = 0\,,$$

it follows, provided $a_1 \neq \beta_1$ and $a_2 \neq \beta_2$, that $\mathrm{V}u\bar{v}$ and $\mathrm{V}\bar{v}u$ are numerical multiples of $\mathrm{V}\beta_1 a_1$ and $\mathrm{V}a_2\beta_2$ respectively, and (assuming $\mathrm{T}u = \mathrm{T}v = 1$) we may write

$$u\bar{v} = \cos\phi \pm \delta_1 \sin\phi \qquad (\delta_1 = \mathrm{UV}\beta_1\alpha_1),$$

$$\bar{v}u = \cos\psi \pm \delta_2 \sin\psi \qquad (\delta_2 = \mathrm{UV}\alpha_2\beta_2).$$

We may also assume

$$\beta_1\alpha_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

$$\alpha_2\beta_2 = -\cos\epsilon_2 + \delta_2 \sin\epsilon_2.$$

Then

$$\mathrm{S}(1 + \beta_1\alpha_1)u\bar{v} = 2\sin\frac{\epsilon_1}{2}\sin\left(\frac{\epsilon_1}{2} \pm \phi\right),$$

$$\mathrm{S}(1 + \alpha_2\beta_2)\bar{v}u = 2\sin\frac{\epsilon_2}{2}\sin\left(\frac{\epsilon_2}{2} \pm \psi\right),$$

and, therefore, in order that the condition $\mathrm{S}u\bar{v} \neq \mathrm{S}\alpha_1 u\beta_1 v$ may fail it is necessary and sufficient that

$$\epsilon_1 = 2n_1\pi \text{ or } 2n_1\pi \pm \phi,$$

and

$$\epsilon_2 = 2n_2\pi \text{ or } 2n_2\pi \pm \psi,$$

or, in equivalent terms,

$$\beta_1\alpha_1 = -1 \text{ or } -(v\bar{u})^2 \text{ or } -(u\bar{v})^2,$$

and

$$\alpha_2\beta_2 = -1 \text{ or } -(\bar{u}v)^2 \text{ or } -(\bar{v}u)^2,$$

and there are nine combinations. Any combination of the type $\beta_1\alpha_1 = -(v\bar{u})^2$, $\alpha_2\beta_2 = -(\bar{u}v)^2$, however, leads to results identical with those obtained from $\beta_1\alpha_1 = -1$, $\alpha_2\beta_2 = -1$; for, from any of the pairs of equations of this type and from § 23 (6) follows

$$\mathrm{S}(u\bar{v})^2 = -\tfrac{1}{2}\mathrm{S}(\alpha_1\beta_1 + \alpha_2\beta_2) = \mathrm{S}^2 u\bar{v},$$

an interchange of factors under S being here permissible; thus $\mathrm{V}u\bar{v} = 0$, $u = \pm v$, and in fact

$$\beta_1\alpha_1 = -1, \quad \alpha_2\beta_2 = -1.$$

But also, this combination is a particular case (obtained by making $u = \pm v$) of either of the remaining two and thus the two alternative conditions

$$\beta_1 = \alpha_1, \quad \beta_2 = \alpha_2(\bar{u}v)^2 \text{ or } \alpha_2(\bar{v}u)^2,$$

and

$$\beta_2 = \alpha_2, \quad \beta_1 = (v\bar{u})^2\alpha_1 \text{ or } (u\bar{v})^2\alpha_1,$$

account for all the cases that can arise. The specific value of $u\bar{v}$ is here undetermined and may vary from a unit scalar to a unit vector value.

It has been tacitly assumed that α_1, β_1 in the one case, and α_2, β_2 in the other, are both essentially positive, but it is evident that the algebraic signs in the equations of the two planes can always be so disposed that this shall be the fact. The conclusion, in its general form, is:

(1) *In order that the two planes* $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may cease to have maximal and minimal isoclinal angles it is necessary that* $\beta_1 = \pm a_1$, *or* $\beta_2 = \pm a_2$ *and any one of these conditions is sufficient.*

(2) *The isoclinal angle* $(a_1 u, \beta_1 v)$ *is constant under all variations of* θ; this is apparent in the equations

$$Sa_1^\theta u\bar{v}\beta_1^{-\theta} = Su\bar{v}, \quad Sua_2^{-\theta}\beta_2^\theta\bar{v} = Su\bar{v},^*$$

the first of which is a consequence of $\beta_1 = a_1$, the second of $\beta_2 = a_2$.

26. *An Example.* The following example illustrates the failure of the conditions for maximal and minimal isoclinal angles. Suppose the equations of the two planes to be

$$a\rho + \rho a = 0, \quad a\rho + \rho\beta = 0.$$

The first is equivalent to

$$2aS\rho + 2SaV\rho = 0,$$

which involves the two equations

$$S\rho = 0, \quad SaV\rho = 0.$$

Hence the values of ρ that satisfy the first equation represent vectors perpendicular to a and may be written in the form

$$\rho = xa^\theta\gamma,$$

in which x and θ are scalar variables and γ is a vector constant, and the condition $Sa\gamma = 0$ is necessary.

The general values of ρ that satisfy the second equation may have the form

$$\rho = xa^\theta(a - \beta).$$

If now $u = \gamma$ and $v = a - \beta$, then

$$Sau\bar{v} = Sa\gamma(\beta - a),$$

which vanishes if $Sa\gamma\beta = 0$, that is, if γ be chosen coplanar with a and β. Hence, with this choice of γ made, the directors γ and $a - \beta$ satisfy the first two conditions for a maximal or a minimal isoclinal angle. But

$$Sa^\theta\gamma(\beta - a)a^{-\theta} = S\gamma(\beta - a) = Su\bar{v}$$

for all values of θ and the third condition fails. The isoclinal angle $(a^\theta u, a^\theta v)$ remains unchanged during any variations of θ. In other words, every line in either plane has the same inclination to the other plane.

* Note that $a_1^\theta u = ua_2^{-\theta}$, $\beta_1^\theta v = v\beta_2^{-\theta}$, and therefore $Sa_1^\theta u\bar{v}\beta_1^{-\theta} = Sua_2^{-\theta}\beta_2^\theta\bar{v}$.

The angle of this inclination (constant isoclinal angle) is represented at C, in the accompanying figure. The equation

$$ap + pa = 0$$

represents a plane through O perpendicular to a, lying wholly within vector space (a space containing only vectors), and

$$ap + p\beta = 0$$

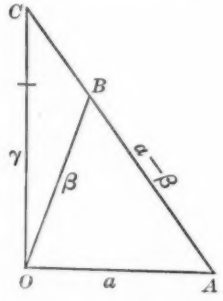
is the equation of a plane that intersects this vector space in a straight line through O parallel to $a - \beta$.

§§ 28, 29. Some Special Cases.

28. *Perpendicularity.* If the maximal isoclinal angle, say $\angle(a_1u, \beta_1v)$, be a right angle, then not only $Sa_1u\bar{v} = 0$, but also $Sa_1u\bar{\beta}_1v = 0$, and

$$Sa_1u\bar{v}\beta_1^{-\theta} = 0$$

for all values of θ; that is, a_1u, a director in one of the planes, is perpendicular to $\beta_1^{\theta}v$, any director (through their meeting point) in the other. This is perpendicularity according to the ordinary definition, though not of the ordinary sort along a line of intersection, for the planes may not meet in a line.



Conversely, if the two planes be perpendicular to one another it must be possible to assign in either of them a director that shall be perpendicular to every director in the other, say a_1u and β_1v, such that

$$Sa_1u\bar{v}\beta_1^{-y} = 0, \quad Sa_1^{x}u\bar{v}\beta_1 = 0,$$

where x and y are arbitrary scalars; and it follows (by assuming in succession $x, y = 0, 1$) that

$$Sa_1u\bar{v} = S\beta_1u\bar{v} = S\beta_1a_1u\bar{v} = 0.$$

These conditions make $\angle(a_1u, \beta_1v)$ a maximal isoclinal angle if $Su\bar{v} \neq 0$.

But $S\beta_1 a_1 u\bar{v}$ is a factor in $S(a_1\beta_1 + a_2\beta_2)$ and therefore:

In order that the planes $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *may satisfy the ordinary definition of perpendicularity it is necessary and sufficient that*

$$S(a_1\beta_1 + a_2\beta_2) = 0.$$

There are three further distinct criteria to be considered, giving rise to three kinds of perpendicularity.

(1) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} \neq 0$ or 1. Resume the equations of § 26 for $u\bar{v}$ and $\beta_1 a_1$, omitting the ambiguous sign not here needed; we have

$$u\bar{v} = \cos\phi + \delta_1 \sin\phi, \quad \beta_1 a_1 = -\cos\epsilon_1 + \delta_1 \sin\epsilon_1,$$

and therefore

$$Sa_1 u\overline{\beta_1 v} = -\cos(\epsilon_1 + \phi) = 0,$$

$$\epsilon_1 + \phi = \frac{\pi}{2}.$$

Hence $Sa_1\beta_1 \neq Sa_2\beta_2$ *and the planes meet only in a point.* According as $\angle(\beta_1, a_1)$ is large or small the minimal isoclinal angle is small or large. The condition may be stated in the form:

$$Sa_1\beta_1 = -Sa_2\beta_2 \neq 0, \text{ or } \pm 1.$$

(2) $Sa_1 u\overline{\beta_1 v} = 0$, $Su\bar{v} = 1$. Here $\phi = 0$, $\epsilon_1 = \pi/2$, and $\beta_1 a_1$ is a vector; hence

$$Sa_1\beta_1 = Sa_2\beta_2 = 0,$$

is the condition necessary and sufficient for simple perpendicularity along a line of intersection (the ordinary kind of perpendicularity); provided also, the planes being $(a_1, a_2, 2a)$, $(\beta_1, \beta_2, 2b)$, the further condition

$$a_1 b - b a_2 + \beta_1 a - a\beta_2 = 0$$

is satisfied (§§ 20, 21).

(3) $Sa_1 u\beta_1 v = Su\bar{v} = 0$. Here $u\bar{v} + v\bar{u} = 0$, $(u\bar{v})^2 = -1$, and therefore

$$\beta_1 a_1 = \pm 1, \quad a_2\beta_2 = \mp 1,$$

that is, $\beta_1 = \mp a_1$ and $\beta_2 = \pm a_2$, the combinations of signs being either $-+$ or $+-$.

If either of these pairs of conditions be assigned, that is, if u' and v' satisfy the equations

$$a_1 u' + u' a_2 = 0, \quad a_1 v' - v' a_2 = 0,$$

we may write (§ 14)

$$u' = a_1^x (a_1 - a_2), \quad v' = a_1^y (a_1 + a_2),$$

in which x and y are arbitrary scalars, and then

$$Su'\bar{v}' = -Sa_1^{x-y}(a_1 a_2 - a_2 a_1),$$

and this is zero for all values of x and y. The geometrical interpretation of this result is that *every line in either plane is perpendicular to every line in the other.* In other words:

The necessary and sufficient conditions for the hyperperpendicularity of the planes $(\alpha_1, \alpha_2, 2a)$, $(\beta_1, \beta_2, 2b)$ *are*

$$\beta_1 = \pm \alpha_1, \quad \beta_2 = \mp \alpha_2;$$

or, expressed in another form,

$$S\alpha_1\beta_1 = -S\alpha_2\beta_2 = \pm 1.$$

The conditions for maximal and minimal isoclinal angles fail.

29. *Parallelism.* If two planes $(\alpha_1, \alpha_2, 2a)$, $(\beta_1, \beta_2, 2b)$ be parallel, the pairs of directors $\alpha_1^\theta u$, $\beta_1^\theta v$* that form their isoclinal angles are parallel for all values of θ, and in particular, if $Tu = Tv = 1$,

$$Su\bar{v} = S\alpha_1 u\beta_1 v = 1,$$

whence

$$S(\alpha_1\beta_1 + \alpha_2\beta_2) = -2.$$

But also, they will have no point of intersection except at an infinite distance, and therefore

$$\frac{\alpha_1 b - b\alpha_2 + \beta_1 a - a\beta_2}{S(\alpha_2\beta_2 - \alpha_1\beta_1)} = \infty,$$

and the further condition $S\alpha_2\beta_2 = S\alpha_1\beta_1$ is necessary. Hence, for the two planes to be parallel it is necessary that

$$S\alpha_2\beta_2 = S\alpha_1\beta_1 = -1.$$

This condition is also sufficient, for, if it is satisfied, it immediately follows that $S(\alpha_1\beta_1 + \alpha_2\beta_2) = -2$ and $Su\bar{v} = S\alpha_1 u\beta_1 v = 1$. Therefore:

(1) *In order that* $(\alpha_1, \alpha_2, 2a)$ *and* $(\beta_1, \beta_2, 2b)$ *may be parallel it is necessary and sufficient that*

$$\alpha_1 = \beta_1, \quad \alpha_2 = \beta_2.$$

(It is understood that the signs before α_1, α_2 in their respective equations are both positive.)

It is immediately evident that the condition

$$\alpha_1 b - b\alpha_2 + \beta_1 a - a\beta_2 = 0$$

is satisfied. This I interpret as meaning that two parallel planes meet in a straight line at infinity.

* These directors are supposed to be placed in their respective planes without reference to a particular origin.

(2) When the conditions for parallelism are satisfied,

$$a_1 a - a a_2 = 0, \quad a_1 b - b a_2 = 0,$$

and we may write

$$a = m a^{\theta}(a_1 + a_2), \quad b = n a_1^{\phi} a,$$

where m, n, θ, ϕ are arbitrary scalars, and the equation of $(\beta_1, \beta_2, 2b)$ becomes

$$a_1 p + p a_2 + 2 n a_1^{\phi} a = 0.$$

It represents a doubly infinite series of planes parallel to $(a_1, a_2, 2a)$, one series being obtained by varying n, the other by varying ϕ.

(3) When the point of intersection of a pair of planes is moved to an infinite distance, they have in general at the limit a point of intersection at infinity. But if the conditions for maximal and minimal isoclinal angles fail these angles become ultimately zero and the conditions for parallelism are satisfied. *Thus planes that have a constant isoclinal angle and meet at infinity are parallel and meet in a straight line.*

§§ 29–32. Isoclinal Systems.

30. *Two-dimensional Systems.* The following group of theorems (§§ 30–32) restate for planes through a point in four-dimensional space the fundamental parts of Clifford's theory of parallels in elliptic space of three dimensions (Hathaway's theory of parallel and contra-parallel great circles on the hypersphere).* The planes and lines here considered are, without exception, supposed to pass through the origin; and by any plane, or any line, is meant a plane or a line satisfying this condition.

Consider a series of planes meeting in a point no two of which have maximal and minimal isoclinal angles, and place the origin at their point of meeting. Corresponding to the two conditions $\beta_1 = \pm a_1$, $\beta_2 = \pm a_2$ of § 26, there are two systems of planes: an a-system whose equations may be written in the form (§ 13)

$$ap - p a^{-1} a a = 0,$$

and a β-system whose equations similarly constructed are

$$b\beta b^{-1} p - p\beta = 0,$$

where a and b are arbitrary versors whose different values produce the several

* It was Professor Hathaway's paper on *Quaternions as Numbers of Four-Dimensional Space* that forced upon my attention the essential identity of the two theories. It should be remarked that here, as also in Hathaway's paper, the theory is stated as applying to elliptic space in its antipodal (spherical) form.

The references to Clifford, Buchheim, Klein, Whitehead are here important.

individual planes of the systems. Any system is thus determined by a characteristic vector α, or β, and by the corresponding form of its equations.

(1) *Any two planes of a system have a unique (constant) isoclinal angle* (§ 23), a consequence of the failure of the conditions for maximal and minimal isoclinal angles (§ 26 (2)).

(2) *A transversal meeting any two planes of a system in straight lines makes equal dihedral angles with them.* For, the two planes being respectively $(\alpha, -\alpha^{-1}a\alpha, 0)$, $(a, -\alpha'^{-1}a\alpha', 0)$ and their transversal being $(\gamma, -\gamma', 0)$, the conditions for straight-line intersections are

$$S\alpha\gamma = S\alpha^{-1}a\alpha\gamma' = S\alpha'^{-1}a\alpha'\gamma',$$

and in either case the cosine of the dihedral angle is $-S\alpha\gamma$ (§ 19). Here, the signs of the scalar functions being alike, the equal dihedrals are the alternate exterior-interior angles on the same side of the transversal.

Thus all the planes of a system determined by any unit vector are equally inclined to any transversal meeting them in straight lines. They constitute what may be called an *isoclinal system*, or a *system of isoclines*. Two planes may be said to be mutually isoclinal when their isoclinal angle is constant; and from this definition it immediately follows that *any two planes that are isoclinal to a third plane are isoclinal to each other.*

(3) *To a given plane through a given straight line there exist always two isoclines.* For, given a plane $(\alpha, -\beta, 0)$ and a straight line ε, both of the planes whose equations are

$$\alpha\rho - \rho\varepsilon^{-1}a\varepsilon = 0, \quad \varepsilon\beta\varepsilon^{-1}\rho - \rho\beta = 0$$

are isoclinal to $(\alpha, -\beta, 0)$, and they intersect in ε.

(4) Given α and a, the equation $a\rho - \rho\alpha^{-1}a\alpha = 0$ is uniquely determined; but since we have identically

$$\alpha^{-1}a\alpha = (\alpha^{\psi}a)^{-1}\,a\,(\alpha^{\psi}a),$$

any pair of values α, $\alpha^{\psi}a$ determines this same plane. Let $a = \beta^{\phi}$; the equation of the planes of the α-system has then the form

$$a\rho - \rho(\alpha^{\psi}\beta^{\phi})^{-1}\,a\,(\alpha^{\psi}\beta^{\phi}) = 0.$$

Changes in ψ give no new planes, but there is a distinct plane for each value of ϕ (within the limits 0, 4), and some (not all) of the differing values of β correspond to different planes.

If a change from β to β_1 makes no effective change in the equation, then

$$\beta_1^{-\phi}a\beta_1^{\phi} = \beta^{-\phi}a\beta^{\phi}, \quad a\beta_1^{\phi}\beta^{-\phi} = \beta_1^{\phi}\beta^{-\phi}a,$$

from which it follows that $V\beta_1^{\phi}\beta^{-\phi}$ is parallel to α; then

$$\beta_1^{\phi} = a^z \beta^{\phi} .$$

Hence, corresponding to the varying scalar values of z, there is a singly infinite series of changes in β which give rise to no new planes.

But any unit vector is determined by two independent scalar parameters and thus one parameter in β is at our disposal for producing new planes of the system. Therefore:

A system of isoclines of given type (an a-system or a β-system), corresponding to a given vector, consists of a doubly infinite series of planes. It is a two-dimensional system.

31. *Cardinal and Ordinal Systems.** For brevity write $a_{\phi} = \beta^{-\phi} a \beta^{\phi}$ and define the two-dimensional system of a-isoclines by the equation

$$ap - pa_{\phi} = 0 .$$

Changes in ϕ produce a singly infinite series of planes whose isoclinal angles, formed by the successive members of the series with a fixed member of it, are all different; for the squared cosine of the isoclinal angle formed by $(a , -a_{\phi'} , 0)$, regarded as fixed, and $(a , -a_{\phi} , 0)$, any other plane of the series, is by § 23 (6)

$$\mathrm{S}^2 u\bar{v} = \tfrac{1}{2}(1 - \mathrm{S}a_{\phi} a_{\phi'}) ,$$

and this varies with ϕ.

Now an infinite series of a-planes which form with a fixed member of the a-series the same (constant) isoclinal angle may be determined in the following manner:

Subject the variable p in the equation $ap - pa_{\phi} = 0$ to the rotational operation $a^{\theta}(\,)a_{\phi'}^{-\theta}$. This leaves undisturbed all the points of the fixed plane $(a , -a_{\phi'} , 0)$ and rotates, without distortion, all other configurations in our four-dimensional space through the angle $\pi\theta$.† The equation is transformed into

$$aa^{\theta} p a_{\phi'}^{-\theta} - a^{\theta} p a_{\phi'}^{-\theta} a_{\phi} = 0 ,$$

and p now satisfies the equation

$$ap - pa_{\phi'}^{-\theta} a_{\phi} a_{\phi'}^{\theta} = 0 ,$$

which belongs to the a-system; and the isoclinal angle formed by the plane represented by this equation, whose variations depend on θ, with the fixed plane $(a , -a_{\phi'} , 0)$ is the same for all values of θ; for its squared cosine is

$$\mathrm{S}^2 u\bar{v} = \tfrac{1}{2}(1 - \mathrm{S}a_{\phi} a_{\phi'}) \qquad (\S 23\,(6)) .$$

Each new value of θ (within the limits 0, 4) produces a new plane.

* So far as I know the characterization of one-dimensional systems of isoclines (parallels in elliptic space) as of two types, here called cardinal and ordinal, has been made in no previous investigation.

† STRINGHAM: loc. cit. (2) (1884), pp. 55–56. Compare also §1 of this paper and HATHAWAY: loc. cit. (3) (1898), pp. 93–94.

The doubly infinite series of a-isoclines is hereby determined. The planes of the system produced by the variations of ϕ are transmuted into one another by successive changes of the isoclinal angles (all different) which they form with a fixed member of the series; I call them *ordinals*. The planes of the system produced by the variations of θ pass into one another by rotations about the fixed plane $(a, -a_{\phi'}, 0)$, with which they form the same isoclinal angle; I call them *cardinals*. Thus the two-dimensional a-system is made up of one-dimensional ordinal and cardinal systems; and the same remark is also obviously true of a β-system.

Evidently any a-isocline belongs both to an ordinal system and to a cardinal system, but two planes in a system of either type belong to different systems of the other type. We may exhaust the doubly infinite series of a-isoclines by setting up an infinite number of systems of either type; for example, a fixed value of ϕ determines a cardinal system and the variations of θ produce the planes of this system; then a second value of ϕ determines a second cardinal system, and so on. Briefly expressed:

A complete system of isoclines consists indifferently of a simply infinite set of ordinal systems, or of a simply infinite set of cardinal systems.

I call attention to the forms of the expressions for p that satisfy respectively the two equations: namely, for the ordinal system $a^x\beta^\phi$, and for the cardinal system $a^y(a + a_\phi)a^\theta_{\phi'}$, where x, y are arbitrary scalar variables.*

32. *Conjugate Systems.* In general, a plane arbitrarily placed does not meet the isoclines of a system in straight lines.

(1) *If a plane have straight-line intersections with three members of a one-dimensional system (ordinal or cardinal) it meets them all in straight lines.*

We may assign as the mono-parametric equation of the system

$$ap - p\lambda_\phi \doteq 0 \qquad (\lambda_\phi = \mu^{-\phi}\lambda\mu^\phi),$$

where λ and μ are constant unit vectors which may be so determined as to fit the case of either an ordinal or a cardinal system. The cutting plane being $(\gamma, -\gamma', 0)$, the condition is that $S\gamma a = S\gamma'\lambda_\phi$ for three different values of ϕ. This makes γ' the axis of a cone on whose surface the three vectors λ_{ϕ_i} $(i = 1, 2, 3)$ lie. But μ is also the axis of this cone and therefore $\gamma' = \mu$ for all values of ϕ, and the condition becomes

$$S\gamma a = S\gamma'\lambda_\phi = S\lambda\mu;$$

and to make $S\gamma a = S\lambda\mu$ it suffices to determine γ by the condition that its angle with a shall be equal to the angle of λ, μ. This condition being assigned write $\gamma = a^\psi\beta a^{-\psi}$, so that $S\gamma a = Sa\beta = S\lambda\mu$. The transversal plane is now an isocline of the ψ-μ-system

* Cf. HATHAWAY: loc. cit. (1) (1897), p. 55.

$$\beta_\psi p - p\mu = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}),$$

every member of which meets in straight lines all the planes of the a-ϕ-system. Thus (1) is proved not merely for one plane but for an entire one-dimensional system.

(2) *No two planes of the same system intersect in a straight line;* for $Sa^2 \neq S\lambda_\phi \lambda_{\phi'}$, unless $\phi = \phi'$.

The two systems are, in the proper sense, *conjugate* to one another.

Note that one plane, $(a, -\mu, 0)$ is isoclinal to all the planes of both systems.

(3) *The corresponding exterior-interior dihedral angles formed at their intersections by the planes of two conjugate systems are all equal to one another;* for (see also § 30 (2)), independently of the values of ϕ and ψ,

$$Sa\beta_\psi = Sa\beta = S\lambda\mu = S\mu\lambda_\phi .$$

(4) *Conjugate Systems of Ordinals.* By assigning $\lambda = a$ and $\mu = \beta$ we obtain at once, as the equations of the conjugate systems of ordinals

$$ap - pa_\phi = 0 \qquad (a_\phi = \beta^{-\phi} a \beta^\phi),$$

$$\beta_\psi p - p\beta = 0 \qquad (\beta_\psi = a^\psi \beta a^{-\psi}).$$

(5) *Conjugate Systems of Cardinals.* If any three planes of the cardinal system,

$$ap - pa_{\phi'}^{-\theta} a_\phi a_{\phi'}^{\theta} = 0,$$

be met by $(\gamma, -\gamma', 0)$ in straight lines we must have

$$S\gamma a = S\gamma' a_{\phi'}^{-\theta} a_\phi a_{\phi'}^{\theta},$$

for three distinct values of θ; whence follows $\gamma' = a_{\phi'}$ and then

$$S\gamma a = Sa_\phi a_{\phi'} = Saa_{\phi - \phi'}.$$

For the determination of γ it suffices to write $\gamma = a^\epsilon a_{\phi-\phi'} a^{-\epsilon}$. The equation of the conjugate system of cardinals then takes the form

$$a^\epsilon a_{\phi-\phi'} a^{-\epsilon} p - pa_{\phi'} = 0.$$

If the equation of the initial system be

$$\beta_{\psi'}^{\epsilon} \beta_\psi \beta_{\psi'}^{-\epsilon} p - p\beta = 0,$$

the corresponding equation of the conjugate system is

$$\beta_{\psi'} p - p\beta^\theta \beta_{\psi - \psi'} \beta^{-\theta} = 0.$$

32. *The Transition to Elliptic Space.* The foregoing theory of isoclinal systems is clearly a three-dimensional geometry with planes as elements; it is, in fact, the geometry of the sheaf of planes in four dimensional space. All of these planes meet the hypersphere in great circles, the straight lines of an elliptic space (antipodal) whose aggregate is a "space of lines." Hence in order to

translate the propositions of isoclinal systems into their equivalents in elliptic space we have merely to take note of the following dualistic correspondences:

In parabolic four-dimensional space:	In elliptic three-dimensional space:
straight lines	~ points,
planes	~ straight lines,
isoclines	~ parallels,
dihedral angles	~ plane angles,
isoclinal angle of two planes	~ perpendicular distance between two straight lines.

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